An introduction to numerical continuation with AUTO

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AUTO

Software for continuation and bifurcation problems in ordinary differential equations.

- Originally developed by Eusebius Doedel, with contributions from many others.
- For more information and to download AUTO visit http://cmvl.cs.concordia.ca/auto [lecture notes and manual]
- For more info on the theory behind continuation [Krauskopf, Osinga & Galán-Vioque (Eds.) Springer, 2007]

Related software includes: DSTool, PyDSTool, XPPAUT, Content, MatCont, and DDE-BifTool.
ODE overview

\[ u'(t) = f(u(t), \lambda), \quad f, u \in \mathbb{R}^n, \quad \lambda \in \mathbb{R} \]

- Compute families of stable and unstable periodic solutions
- Locate and continue folds, branch points, period doubling bifurcations, and bifurcations to tori.
- Do each of the above for rotations
- Follow homoclinic orbits and detect and continue codim-2 bifurcations, using HomCont.
- Compute curves of solutions on \([0,1]\), subject to nonlinear boundary and integral conditions.
- Determine folds and branch points along solution families to the above BVP.

and more...

[Creaser et al. 2017]
Predator prey model

Consider a predator-prey model with fishing

\[
\begin{align*}
    u_1' &= 3u_1(1 - u_1) - 2u_1u_2 - \lambda(1 - e^{-5u_1}), \\
    u_2' &= -u_2 + 3u_1u_2.
\end{align*}
\]

We can think of \( u_1 \) as ‘fish’ and \( u_2 \) as ‘sharks’, while the term

\[ \lambda(1 - e^{-5u_1}), \]

represents ‘fishing’ with a ‘quota’ \( \lambda \).

For \( \lambda = 0 \) there are three equilibria

\[
(u_1, u_2) = (0, 0), (1, 0), \left( \frac{1}{3}, 1 \right).
\]
**Implicit Function Theorem**

Consider $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ with $f(u_0, \lambda_0) = 0$ for $u_0 \in \mathbb{R}^n$ and $\lambda_0 \in \mathbb{R}$. Suppose the following holds:

- The Jacobian matrix $f_u(u_0, \lambda_0)$ is nonsingular;
- $f$ and $f_u$ are Lipschitz continuous (in both $u$ and $\lambda$)

Then there exists $\delta > 0$ and interval $\Lambda_\delta = (\lambda_0 - \delta, \lambda_0 + \delta)$, with a unique function $u(\lambda)$ continuous on $\Lambda_\delta$, such that

$$u(\lambda_0) = u_0$$
$$f(u(\lambda), \lambda) = 0, \text{ for all } ||\lambda - \lambda_0|| < \delta.$$  

*We call $u_0$ an isolated solution of $f(u, \lambda_0) = 0$.***
**JACOBIANS**

The Jacobian matrix for our example is

\[
J(u_1, u_2; \lambda) = \begin{pmatrix}
3 - 6u_1 - 2u_2 - 5\lambda e^{-5u_1} & -2u_1 \\
0 & -1 + 3u_1
\end{pmatrix}.
\]

Evaluating at the equilibria gives

\[
J(0, 0; 0) = \begin{pmatrix}
3 & 0 \\
0 & -1
\end{pmatrix}, \quad J(1, 0; 0) = \begin{pmatrix}
3 & -2 \\
0 & 2
\end{pmatrix}, \\
\]

\[
J\left(\frac{1}{3}, 1; 0\right) = \begin{pmatrix}
-1 & -\frac{2}{3} \\
6 & 0
\end{pmatrix}
\]

all three are non-singular for \( \lambda = 0 \). Therefore, by the IFT all three equilibria persist for (small) \( \lambda \neq 0 \).
PHASE PORTRAITS

$\lambda = 0$

$\lambda = 0.5$

$\lambda = 0.68$

$\lambda = 0.75$
**Parameter continuation**

- Set $\lambda_1 = \lambda_0 + \Delta \lambda$
- Use Newton’s method with initial guess $u_1^{(0)} = u(\lambda) + \Delta \lambda \dot{u}$
Pseudo-arclength continuation

- $f(u_1, \lambda_1) = 0$
- $(u_1 - u_0)\dot{u}_0 + (\lambda_1 - \lambda_0)\dot{\lambda}_0 - \Delta s = 0$
USER SUPPLIED FILES

To run AUTO you will need:

- A source file \texttt{xxx.\{f,f90,c\}} containing the Fortran routines \texttt{FUNC}, \texttt{STPNT}, \texttt{BCND}, \texttt{ICND}, \texttt{FOPT}, and \texttt{PVLS}.

- A constants-file \texttt{c.xxx}.

See auto/07p/demos for lots of examples.

AUTO is run via the AUTO command line user interface (CLUI) - based on Python
AUTO Errors

Error MX

- Something in c. file? Problem with parameters, not initialised properly (not good maths!) or doesn’t quite match BCs, check PVLS and BCs.

Error NaN

- Something in .f90 file?

Error ValueError: invalid literal for int() with base 10: 'x-

- Missed a comma in the c.file

Error IndexError: string index out of range

- Some typo in your c. file - go back to an older one and try again.

Error write() argument must be str, not bytes

- in c. file if IRS greater than 0 then you get this error if using python3