# Stability index for a riddled basin of attraction for a piecewise linear map

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**Objectives:** Prove analytically for a specific example

- Existence of riddled basin
- Calculations of stability index

**Outline:**

- Piecewise linear (PWL) skew product map
- Boundary between basins
- Prove existence of riddled basin
- Stability index (point & attractor)
- Non-convergence of stability index

**Motivations**

- Understanding riddled basins of attraction.
- Using stability index to quantify local geometry of riddled basin of attractor.
- The results on stability index for our PWL map inspired by Keller (2014) (we use different method).

**Related works for riddled basins**

- Ott *et al.* (1994).
- Ashwin *et al.* (1998), Ashwin (2005).

# The PWL model

We consider the skew product transformation in the unit square $(\theta, x) \in [0,1]^2$

$$F(\theta, x) = (T_s(\theta), h(\theta, x)) \tag{1}$$

where the chaotic base map

$$T_s(\theta) = \begin{cases} \frac{\theta}{s} & \text{if } 0 \leq \theta < s, \\ (\theta - s)(1-s)^{-1} & \text{if } s < \theta \leq 1, \end{cases} \tag{2}$$

is the skewed doubling map and the fibre map

$$h(\theta, x) = \begin{cases} \min(\gamma x, 1) & \text{if } 0 \leq \theta < s \text{ and } x < 1, \\ \delta x & \text{if } s < \theta \leq 1 \text{ and } x < 1, \\ 1 & \text{if } x = 1, \end{cases} \tag{3}$$

where $0 < s < 1$, $\gamma > 1$, $0 < \delta < 1$. NB $\gamma$ and $\delta$ are expansion and contraction resp. We study the special case $\gamma = 1/\delta$.

- From (3), $x = 0$ and $x = 1$ are invariant sets.
- Thus, we denote

$$A_0 = [0,1] \times \{0\},$$
$$A_1 = [0,1] \times \{1\},$$

where $A_0$ and $A_1$ are disjoint compact invariant sets.

- The basins are

$$B_0 := \mathcal{B}(A_0) = \{(\theta, x) : d(F^n(\theta, x), A_0) \to 0 \text{ as } n \to \infty\},$$
$$B_1 := \mathcal{B}(A_1) = \{(\theta, x) : d(F^n(\theta, x), A_1) \to 0 \text{ as } n \to \infty\}.$$

- $A_0$ and $A_1$ are attractors.

# Basins of attraction

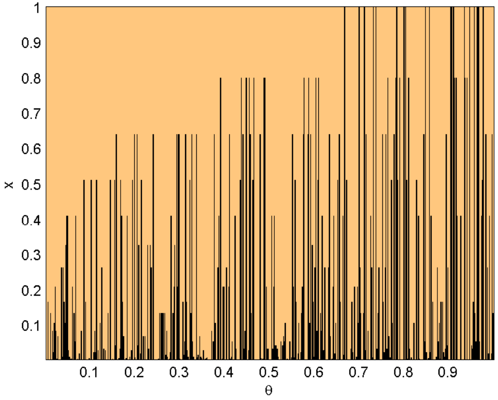



Figure: Numerical approximation of the basins of attraction with $\delta = 0.8$ and $s = 0.49$ for map $F$. Black region is $B_0$ and orange region is $B_1$. **NB** $B_0 \cap B_1 = \emptyset$.

- Since our base map is Markov we can divide $[0,1]^2$ using the following partition:

$$[0,1]^2 = \bigcup_{k=1}^{\infty} X_k$$

where

$$\begin{aligned} X_k &= X_{k,1} \dot{\bigcup} X_{k,2}, \\ X_{k,1} &= [0,s] \times [\delta^k, \delta^{k-1}], \\ X_{k,2} &= [s,1] \times [\delta^k, \delta^{k-1}], \end{aligned}$$

where $\dot{\bigcup}$ denotes the disjoint union.

- Actions of map $F$:

$$F(X_{k,1}) = X_{k-1} \text{ for } k \geq 2, \tag{4}$$
$$F(X_{1,1}) = A_1, \tag{5}$$
$$F(X_{k,2}) = X_{k+1} \text{ for } k \geq 1. \tag{6}$$

# Actions of map $F$

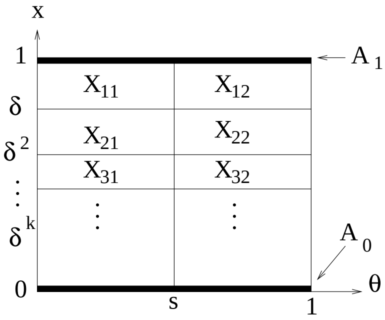



Figure: The schematic diagram for map $F$ (1).

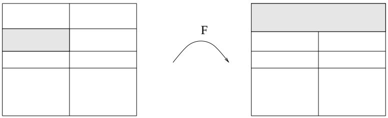



Figure: The effect of $F$ on $X_{k,1}$ for $k \geq 2$.

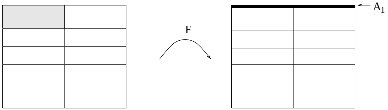



Figure: The effect of $F$ on $X_{1,1}$: $F(X_{1,1}) = A_1$.

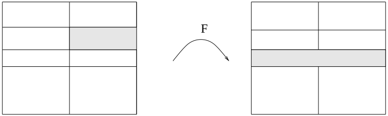



Figure: The effect of $F$ on $X_{k,2}$ for $k \geq 1$.

# Boundary between $B_0$ and $B_1$

- We consider $[s, 1] \subset [0, 1]$ and investigate frequency of an orbit of $\theta \in \mathbb{T}$ under $T_s(\theta)$ visit the right interval.
- Define

$$n_k(\theta) := \begin{cases} 0 & \text{if } T_s^k(\theta) < s, \\ 1 & \text{if } T_s^k(\theta) > s, \end{cases} \tag{7}$$

- Then define # of first $N$ points that the orbit of $\theta$ lies in $[s, 1]$:

$$i_N(\theta) := \sum_{k=0}^{N-1} n_k(\theta) \tag{8}$$

- Frequency with which the orbit of $\theta$ lies in $[s, 1]$:

$$\lim_{N\to\infty} \frac{i_N(\theta)}{N} = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} n_k(\theta) \tag{9}$$

# Boundary between $B_0$ and $B_1$ contd.

- We assume $A_0$ supports an ergodic $T_s$-invariant measure $\mu$.
- For any ergodic measure $\mu \in \mathcal{P}(T_s)$

$$\lim_{N\to\infty} \frac{i_N(\theta)}{N} = \int_s^1 d\mu(\theta) \quad \text{for } \mu - \text{a.a. } \theta. \tag{10}$$

- Example: For Lebesgue measure

$$\lim_{N\to\infty} \frac{i_N(\theta)}{N} = 1 - s \quad \text{for } \ell - \text{a.a. } \theta.$$

**Definition**

*The basin boundary is defined as follows:*

$$\hat{\varphi}_\infty(\theta) = \inf_{N\geq 0} \left\{ \delta^{N-2i_N(\theta)} \right\}.$$

# Riddled basins: definitions

- Introduced by Alexander *et al.* (1992).

**Definition**

*A Milnor attractor $A$ has a riddled basin if for all $x \in \mathcal{B}(A)$ and $\varepsilon > 0$, then*

$$\ell(B_\varepsilon(x) \cap \mathcal{B}(A))\ell(B_\varepsilon(x) \cap \mathcal{B}(A)^c) > 0. \quad (11)$$

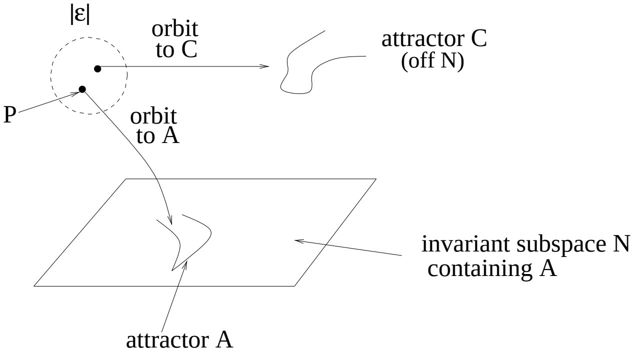
- Specific case:

**Definition**

*The basin of a Milnor attractor $A_0$ is riddled with the basin of a second Milnor attractor $A_1$, if for all $\varepsilon > 0$ and $x \in B_0$,*

$$\ell(B_\varepsilon(x) \cap B_0) > 0 \text{ and } \ell(B_\varepsilon(x) \cap B_1) > 0. \quad (12)$$

# Schematic diagram



# Proofs: existence of a riddled basin $B_0$

**Theorem**

*For any* $0 < \delta < 1$ *and* $s < 1/2$, $B_0$ *is riddled with* $B_1$.

Required 4 steps of proofs:

- The union of $B_0$ and $B_1$ is full measure.

**Theorem**

*For any* $0 < \delta < 1$, $0 < s < 1$, $s \neq 1/2$ *and almost all* $\theta$,

(i) *if* $x_0 < \hat{\varphi}_\infty(\theta)$ *then* $(\theta, x_0) \in B_0$,

(ii) *if* $x_0 > \hat{\varphi}_\infty(\theta)$ *then* $(\theta, x_0) \in B_1$.

*Hence* $\ell(B_0 \cup B_1) = 1$.

# Proofs: existence of a riddled basin $B_0$ contd.

- If $s < 1/2$, both basins have positive measure, i.e. $\ell(B_0) > 0$ and $\ell(B_1) > 0$.
- Show $B_1$ is dense in $[0, 1]^2$.
- Show $B_1$ has positive measure on any neighbourhood in $[0, 1]^2$.

# Stability index $\sigma$

- Introduced by Podvigina and Ashwin (2011).
- To characterize local geometry of basins of attraction for heteroclinic cycles.
- Keller (2014) - for chaotically driven concave maps.
- Our case: characterize local structure of riddled basin in PWL map - for point & attractor

# Stability index at point $(\theta, 0)$

- For a point $\theta \in [0,1]$ and $\varepsilon > 0$, define

$$\Sigma_\varepsilon(\theta) := \frac{\ell(B_\varepsilon(\theta) \cap B_0)}{\ell(B_\varepsilon(\theta))}, \tag{13}$$

  i.e.,

$$1 - \Sigma_\varepsilon(\theta) := \frac{\ell(B_\varepsilon(\theta) \cap B_1)}{\ell(B_\varepsilon(\theta))}, \tag{14}$$

  where $0 \le \Sigma_\varepsilon(\theta) \le 1$.

- Then the stability index at $\theta$ is defined to be

$$\sigma(\theta) := \sigma_+(\theta) - \sigma_-(\theta), \tag{15}$$

  where

$$\sigma_-(\theta) := \lim_{\varepsilon \to 0} \left[ \frac{\log(\Sigma_\varepsilon(\theta))}{\log \varepsilon} \right], \quad \sigma_+(\theta) := \lim_{\varepsilon \to 0} \left[ \frac{\log(1 - \Sigma_\varepsilon(\theta))}{\log \varepsilon} \right],$$

  as long as these limits converge.

$\mathcal{P}(T_s)$ - the set of all ergodic measures for the map $T_s(\theta)$ such that $\mu \in \mathcal{P}(T_s)$.

**Theorem**

*For $s < 1/2$, any $0 < \delta < 1$ and any $\mu \in \mathcal{P}(T_s)$, for $\mu$-almost all $\theta$;*

$$\sigma(\theta, 0) = \begin{cases} \frac{\log \tilde{\delta} - \log \delta}{\log \delta} \cdot \left( \frac{\lambda_{\parallel}(\theta) - \lambda_{\perp}(\theta)}{\lambda_{\parallel}(\theta)} \right) > 0 & \text{if } \lambda_{\parallel}(\theta) - \lambda_{\perp}(\theta) > 0, \\ \left( \frac{\lambda_{\parallel}(\theta) - \lambda_{\perp}(\theta)}{\lambda_{\parallel}(\theta)} \right) < 0 & \text{if } \lambda_{\parallel}(\theta) - \lambda_{\perp}(\theta) < 0, \end{cases} \tag{16}$$

*where $\tilde{\delta} = \frac{2s\delta}{2-2s} < \delta$ and where $\lambda_{\parallel}(\theta)$ and $\lambda_{\perp}(\theta)$ are the Lyapunov exponents in the base direction and fibre direction respectively.*

Express our $\sigma$ in terms of Lyapunov exponents for PWL map $F$.

**Lemma**

*Suppose $t := \int_0^s d\mu(\theta)$. Then the Lyapunov exponent in the base direction is*

$$\lambda_{\|}(\theta) = -t \log s - (1-t)\log(1-s).$$

**Lemma**

*Suppose $t := \int_0^s d\mu(\theta)$. Then the Lyapunov exponent in the fibre direction is*

$$\lambda_{\perp}(\theta) = (1-2t)\log\delta.$$

# Results - for a special case $\mu = \ell^1$

NB $t = \int_0^s d\ell = s$.

**Theorem**

*For $s < 1/2$, any $0 < \delta < 1$ and $\ell^1 \in \mathcal{P}(T_s)$,*

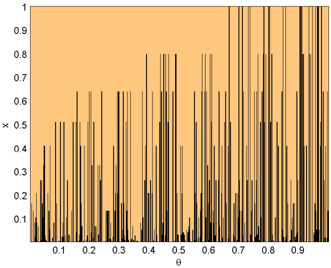
(i) *For $\ell^1$-almost all $\theta$, we have $\theta$ with positive stability index, i.e. $\sigma(\theta, 0) > 0$,*

(ii) *There exists a $\theta$ with negative stability index (i.e. $\sigma(\theta, 0) < 0$) if and only if $\delta < s$.*

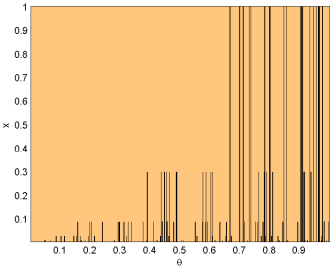
Basins of attraction with corresp. $\sigma$:

(a) $\sigma > 0$ for $\ell^1$-almost all $\theta$.

(b) $\sigma < 0$ for some $\theta \Leftrightarrow \delta < s$.




(a) $\delta = 0.8$ and $s = 0.49$.




(b) $\delta = 0.3$ and $s = 0.49$.

# Results: $\sigma$ vs. parameter $\delta$

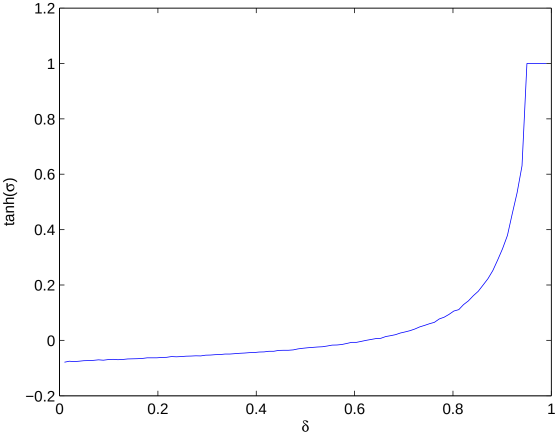



Figure: The stability index $\sigma(\theta, 0)$ over parameter $\delta = 0.01, \ldots, 0.99$ and fixed value $s = 0.49$ for a typical point $\theta = 0.9643$. **NB** $\tanh(\sigma) = 1$ represents the index $\sigma = +\infty$.

# Stability index for attractor $A_0$

- Let $A_0 = [0,1] \times \{0\}$ be an invariant set and let $\varepsilon > 0$. We define

$$\Sigma_\varepsilon(A_0) := \frac{\ell(B_\varepsilon(A_0) \cap B_0)}{\ell(B_\varepsilon(A_0))}, \tag{17}$$

so that

$$1 - \Sigma_\varepsilon(A_0) := \frac{\ell(B_\varepsilon(A_0) \cap B_1)}{\ell(B_\varepsilon(A_0))}, \tag{18}$$

where $0 \leq \Sigma_\varepsilon(A_0) \leq 1$.

- Then the stability index for the invariant set $A_0$ is defined to be

$$\sigma(A_0, B_0) := \sigma_+(A_0) - \sigma_-(A_0), \tag{19}$$

which exists when the following converge:

$$\sigma_-(A_0) := \lim_{\varepsilon\to 0} \frac{\log(\Sigma_\varepsilon(A_0))}{\log \varepsilon}, \quad \sigma_+(A_0) := \lim_{\varepsilon\to 0} \frac{\log(1 - \Sigma_\varepsilon(A_0))}{\log \varepsilon}. \tag{20}$$

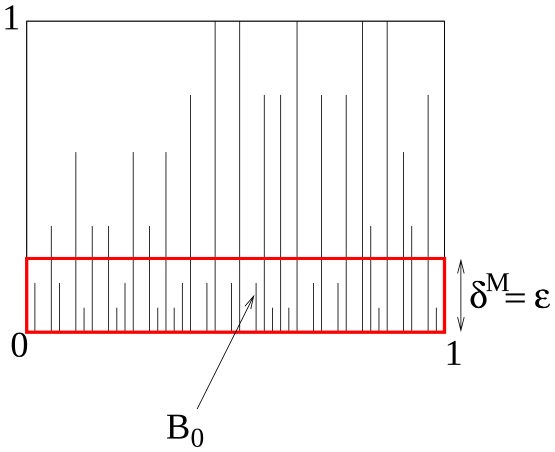



Figure: The schematic diagram showing the neighbourhood of the attractor $A_0 = [0, 1] \times \{0\}$. $B_0$ is the basin for $A_0$.

# Results

## Theorem

*For* $s < 1/2$, *any* $0 < \delta < 1$ *and* $\varepsilon > 0$,

$$\sigma(A_0) = \frac{\log \tilde{\delta} - \log \delta}{\log \delta},$$

*where* $A_0$ *is the attractor at the baseline.*

## Corollary

*For* $s < 1/2$, *any* $0 < \delta < 1$ *and any* $\mu \in \mathcal{P}(T_s)$, *for* $\mu$*-almost all* $\theta$;

$$\sigma(\theta, 0) = \begin{cases} \sigma(A_0) \cdot \frac{\lambda_{\|}(\theta) - \lambda_{\perp}(\theta)}{\lambda_{\|}(\theta)} > 0 & \text{if } \lambda_{\|}(\theta) - \lambda_{\perp}(\theta) > 0, \\ \frac{\lambda_{\|}(\theta) - \lambda_{\perp}(\theta)}{\lambda_{\|}(\theta)} < 0 & \text{if } \lambda_{\|}(\theta) - \lambda_{\perp}(\theta) < 0, \end{cases}$$

*where* $\sigma(A_0)$ *is the stability index of* $A_0$.

# Results: Non-convergence of $\sigma$

**Theorem**

*Suppose $\theta$ is such that*

$$\limsup_{N\to\infty} \frac{i_N(\theta)}{N} \neq \liminf_{N\to\infty} \frac{i_N(\theta)}{N}.$$

*Then $\sigma(\theta, 0)$ will not converge.*

# Summary

- Stability index as a bifurcation tool for riddled basins.
- For the case of riddled basin,
  1. For Lebesgue almost all points in the invariant set, the stability indices are positive.
  2. There may be some points in the invariant set that have negative stability index (iff $\delta < s$).
- Corollary - stability index of a point formulated in terms of Lyapunov exponents and the stability index for a set (Loynes' exponent in Keller's paper).
- However, there are also some points for which the limits of stability indices do no converge.

# Further work

1. The results for the stability indices are restricted to one example of a PWL map.
2. We compared our result with Keller's stability index.
3. Keller's results more general - uses powerful techniques of thermodynamic formalism to obtain $\sigma$.
4. Our interest - to see whether these techniques can be generalized to understand stability index in other (e.g. non-skew product) cases.

## References


- Alexander, J.C., Yorke, J.A., You, Z. and Kan, I., *Riddled basin*, International Journal of Bifurcation and Chaos, 1992, **2**, pp. 795–813.
- Keller, G., *Stability index for chaotically driven concave maps*, Journal of the London Mathematical Society, 2014, **2**, pp. 1–20.
- Ott, E. and Alexander, J.C. and Kan, I. and Sommerer, J.C., *The transition to chaotic attractors with riddled basins*, Physica D, 1994, **76**, pp. 384–410.
- Podvigina, O. and Ashwin, P., *On local attraction properties and a stability index for heteroclinic connections*, Nonlinearity, 2011, **24**, pp. 887–929.
- Lohse, A.S., *Attraction properties and non-asymptotic stability of simple heteroclinic cyccles and networks in* $\mathbb{R}^4$, Universität Hamburg, 2014 (PhD thesis).

# THANKS FOR YOUR ATTENTION!