Heteroclinic Networks: stability, switching and memory

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2. Stability of Heteroclinic Cycles
   - The Guckenheimer–Holmes cycle

3. Heteroclinic Networks
   - The Kirk–Silber network
   - Decision network

4. Summary
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4. Summary
The side-blotched lizard (*Uta stansburiana*) lives in the Pacific Coast Range of California.

There are three types of male lizard, each with a different coloured throat indicating three types of genetically determined mating strategy.
Strategy 1: Have a lot of territory. Orange-throated males establish large territories, with several females. The more females the more often they can mate.

Strategy 2: Be sneaky. Yellow-throated males are sneaky and can mimic the behaviour of females.

Strategy 3: Guard your mate. Blue-throated males defend small territories holding just a few females. Because the territories are so small they can guard their mates carefully.

[Cartoon taken from http://www.sciencenewsforkids.org]
Consider the ODE:

\[ \dot{x} = f(x), \quad x \in \mathbb{R}^n. \tag{1} \]

- An equilibrium \( \xi \) of (1) satisfies \( f(\xi) = 0 \).

- A solution \( \phi_j \) of (1) is a heteroclinic connection from \( \xi_j \) to \( \xi_{j+1} \), if it is backward asymptotic to \( \xi_j \) and forward asymptotic to \( \xi_{j+1} \).

- A heteroclinic cycle is a set of equilibria \( \{\xi_1, ..., \xi_m\} \) and orbits \( \{\phi_1, ..., \phi_m\} \), where \( \phi_j \) is a heteroclinic connection between \( \xi_j \) and \( \xi_{j+1} \), and \( \xi_1 \equiv \xi_{m+1} \).
In systems with invariant subspaces, heteroclinic connections can exist robustly.

Invariant subspaces arise naturally in systems with symmetry.

\( \mathbb{Z}_2 \) symmetry: \( \kappa_3 : (x_1, x_2, x_3) \rightarrow (x_1, x_2, -x_3) \)

\( \text{Fix}(\kappa_3) = \{(x_1, x_2, 0)\} \)
Examples

Population models

Strategy 1: Have a lot of territory. Orange-throated males establish large territories, with several females. The more females the more often they can mate.

Strategy 2: Be sneaky. Yellow-throated males are sneaky and can mimic the behaviour of females.

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Fluid dynamics

N. Becker and G. Ahlers
It can be useful to classify the eigenvalues near each equilibrium.

- Radial
- Contracting
- Expanding
- Transverse
Analysis of flow near heteroclinic cycles

Local flow: $H_1^{\text{in}} \rightarrow H_1^{\text{out}}$
\[
\begin{align*}
\dot{x}_2 &= \lambda_u x_2, \\
\dot{x}_3 &= -\lambda_s x_3.
\end{align*}
\]

Global flow: $H_1^{\text{out}} \rightarrow H_2^{\text{in}}$

Linearise flow around heteroclinic connections.

- Combine local and global flow to get a Poincaré map.
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The Guckenheimer–Holmes cycle

Guckenheimer and Holmes, 1988

The cycle exists in a system with symmetry $\mathbb{Z}_3 \ltimes (\mathbb{Z}_2)^3$.

- Contracting eigenvalue $-\lambda_s$, expanding eigenvalue $\lambda_u$.
- Local map gives $x_2 \to x_3^\delta$, $\delta = \lambda_s / \lambda_u$.
- Global map $x_3 \to Ax_3$. 
The Guckenheimer–Holmes cycle

- Poincaré map: $x \rightarrow Ax^\delta$.
- Fixed points exist at $x = 0$ and $x = A^{1/(1-\delta)}$.
- A resonance bifurcation at $\delta = 1$ produces a long-period periodic orbit.
- Period of orbit $T \sim \frac{1}{1 - \delta}$.
The Guckenheimer–Holmes cycle

- Poincaré map: \( x \rightarrow Ax^\delta \).
- Fixed points exist at \( x = 0 \) and \( x = A^{1/(1-\delta)} \).
- A resonance bifurcation at \( \delta = 1 \) produces a long-period periodic orbit.
- Period of orbit \( T \sim \frac{1}{1 - \delta} \).
Noisy heteroclinic cycles

*Stone and Holmes, 1990*

- Consider additive noise to a heteroclinic cycle.
- Mean passage time past an equilibrium

\[ T \sim \frac{\log \eta}{\lambda_u} \]

where \( \eta \) is the noise amplitude
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A heteroclinic network is a connected union of heteroclinic cycles.

Stability conditions of the network as a whole may be quite complicated.

Both resonance and noise in networks can give complicated behaviour.
Kirk and Silber network

*Kirk and Silber, 1994*
Kirk and Silber network
Subspace dynamics
The position the trajectory hits $H^\text{in},A_B$ determines which equilibrium is visited next.
Kirk and Silber network

Construction of maps

- Plot the Poincare section with polar coordinates.

If the network is attracting trajectories can switch one way but not the other.

If network is not attracting, there can be periodic orbits lying close to either or both of the sub cycles.

*Kirk, Postlethwaite and Rucklidge, SIADS 2012*
Noisy Kirk and Silber network

*Armbruster, Stone and Kirk, 2003*

- Consider additive noise.
- Depending on parameters, the ‘noise ellipse’ can be centered at the origin in $H_{B}^{A}$.
- Proportion of times each cycle visited proportional to shaded area.
For certain parameter sets, noise ellipse can move into basin of attraction of one cycle or the other.

This is termed *lift-off*.

Lift-off can *reduce* switching.
Decision network

*Ashwin and Postlethwaite, 2013*

- Four sub-cycles, each with four equilibria.
- Deterministic case is very complicated: numerics show switching between sub cycles.
- Addition of noise can allow *memory.*
We can choose parameters to have lift-off occur in the $x_3$ direction as the trajectory passes $\xi_5$. 
Red: trajectories which visited $\xi_5$ on the previous loop.
Black, blue, green: trajectories which visited $\xi_6$, $\xi_7$ and $\xi_8$.
Can see lift-off in the $x_3$ direction for red points.
## Transition matrices

<table>
<thead>
<tr>
<th>Without memory:</th>
<th>With memory:</th>
</tr>
</thead>
</table>
| \[
\begin{pmatrix}
0.46 & 0.26 & 0.18 & 0.10 \\
0.47 & 0.23 & 0.20 & 0.10 \\
0.49 & 0.23 & 0.18 & 0.10 \\
0.50 & 0.22 & 0.18 & 0.10 \\
\end{pmatrix}
\] | \[
\begin{pmatrix}
0.63 & 0.33 & 0.03 & 0.02 \\
0.49 & 0.27 & 0.16 & 0.08 \\
0.54 & 0.24 & 0.13 & 0.09 \\
0.52 & 0.27 & 0.14 & 0.07 \\
\end{pmatrix}
\] |
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Heteroclinic cycles and networks can lose stability and produce nearby long-period periodic orbits in resonance bifurcations.

Noisy heteroclinic cycles look like periodic orbits.

Noisy heteroclinic networks can have much more complicated behaviour, including switching and memory.

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