Heteroclinic Networks: stability, switching and memory

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Outline

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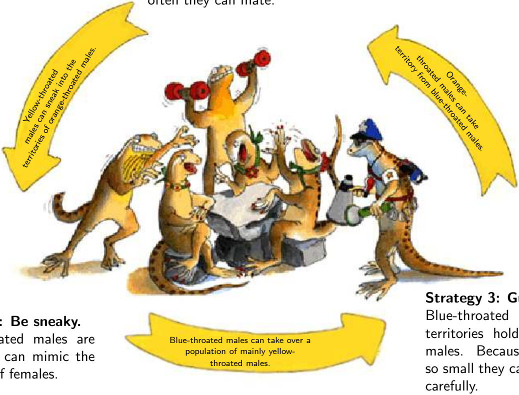
Side-blotched lizard

- The side-blotched lizard (*Uta stansburiana*) lives in the Pacific Coast Range of California.
- There are three types of male lizard, each with a different coloured throat indicating three types of genetically determined mating strategy.

Side-blotched lizard

Strategy 1: Have a lot of territory.
Orange-throated males establish large territories, with several females. The more females the more often they can mate.



Strategy 2: Be sneaky.
Yellow-throated males are sneaky and can mimic the behaviour of females.

Strategy 3: Guard your mate.
Blue-throated males defend small territories holding just a few females. Because the territories are so small they can guard their mates carefully.

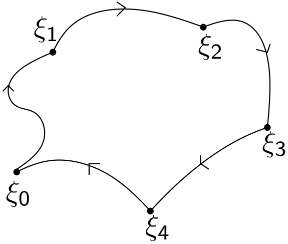
[Cartoon taken from `http://www.sciencenewsforkids.org`]

Heteroclinic cycles

Consider the ODE:

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n. \tag{1}$$

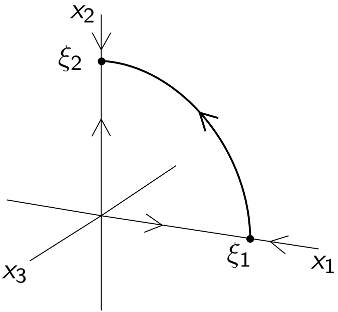
- An equilibrium ξ of (1) satisfies $f(\xi) = 0$.
- A solution ϕ_j of (1) is a heteroclinic connection from ξ_j to ξ_{j+1}, if it is backward asymptotic to ξ_j and forward asymptotic to ξ_{j+1}.
- A heteroclinic cycle is a set of equilibria $\{\xi_1, ..., \xi_m\}$ and orbits $\{\phi_1, ..., \phi_m\}$, where ϕ_j is a heteroclinic connection between ξ_j and ξ_{j+1}, and $\xi_1 \equiv \xi_{m+1}$.



Heteroclinic cycles in systems with symmetry

Field 1980, Krupa and Melbourne 1995, and many others

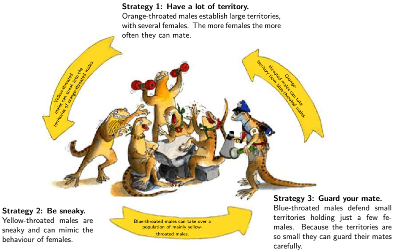
- In systems with invariant subspaces, heteroclinic connections can exist robustly.
- Invariant subspaces arise naturally in systems with symmetry.



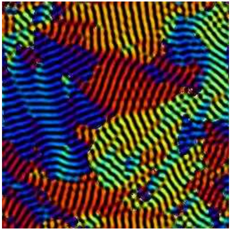
- $\mathbb{Z}_2$ symmetry: $\kappa_3 : (x_1, x_2, x_3) \to (x_1, x_2, -x_3)$
- $\mathrm{Fix}(\kappa_3) = \{(x_1, x_2, 0)\}$

Examples

Population models



Fluid dynamics



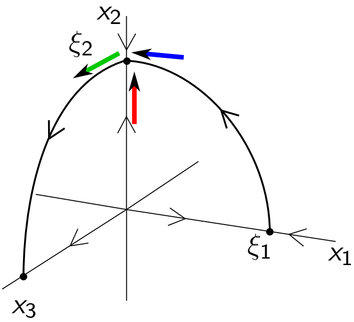
N. Becker and G. Ahlers

Classification of eigenvalues

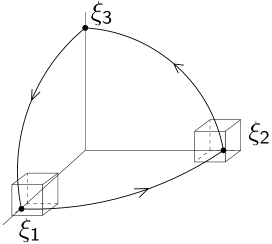
Krupa and Melbourne, 1995

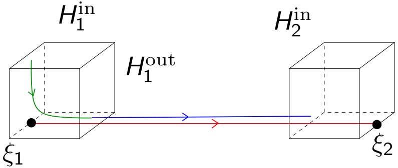
It can be useful to classify the eigenvalues near each equilibrium.

- Radial
- Contracting
- Expanding
- Transverse



Analysis of flow near heteroclinic cycles





Local flow: $H_1^{\mathrm{in}} \to H_1^{\mathrm{out}}$

$$\dot{x}_2 = \lambda_u x_2,$$
$$\dot{x}_3 = -\lambda_s x_3.$$

Global flow: $H_1^{\mathrm{out}} \to H_2^{\mathrm{in}}$

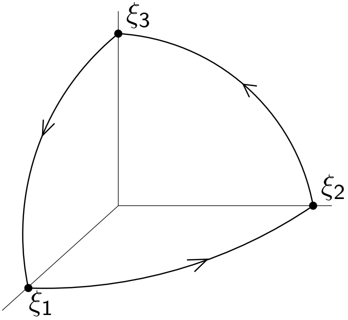
Linearise flow around heteroclinic connections.

- Combine local and global flow to get a Poincaré map.

Outline

The Guckenheimer–Holmes cycle

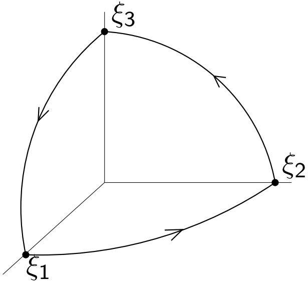
Guckenheimer and Holmes, 1988



The cycle exists in a system with symmetry $\mathbb{Z}_3 \ltimes (\mathbb{Z}_2)^3$.

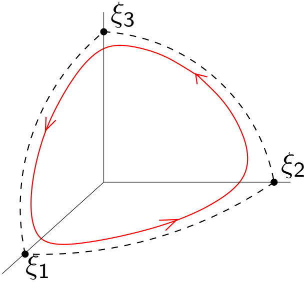
- Contracting eigenvalue $-\lambda_s$, expanding eigenvalue λ_u.
- Local map gives $x_2 \to x_3^\delta$, $\delta = \lambda_s/\lambda_u$.
- Global map $x_3 \to Ax_3$.

The Guckenheimer–Holmes cycle



- Poincaré map: $x \to Ax^{\delta}$.
- Fixed points exist at $x = 0$ and $x = A^{1/(1-\delta)}$.
- A resonance bifurcation at $\delta = 1$ produces a long-period periodic orbit.
- Period of orbit $T \sim \dfrac{1}{1-\delta}$.

The Guckenheimer–Holmes cycle



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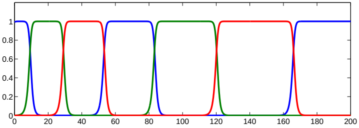
Noisy heteroclinic cycles

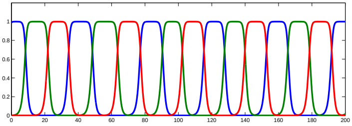
Stone and Holmes, 1990

- Consider additive noise to a heteroclinic cycle.
- Mean passage time past an equilibrium

$$T \sim \frac{\log \eta}{\lambda_u}$$

where η is the noise amplitude

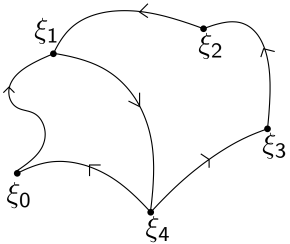




Outline

Heteroclinic networks

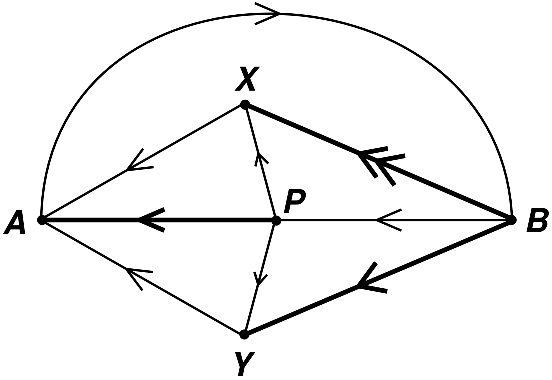
- A heteroclinic network is a connected union of heteroclinic cycles.



- Stability conditions of the network as a whole may be quite complicated.
- Both resonance and noise in networks can give complicated behaviour.

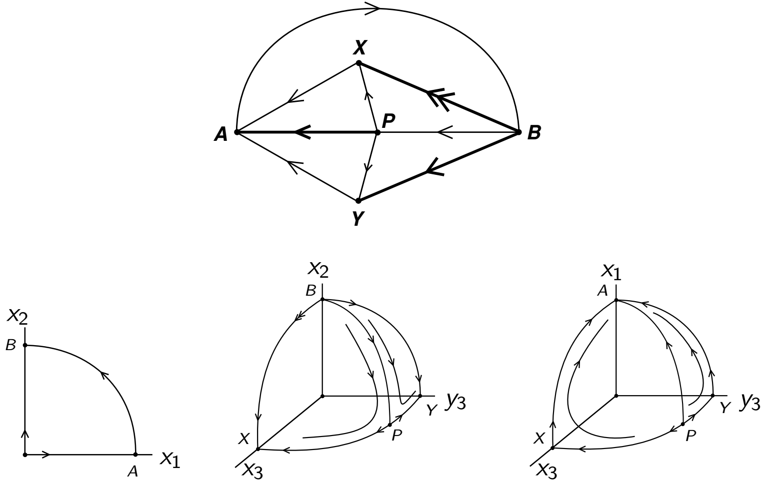
Kirk and Silber network

Kirk and Silber, 1994



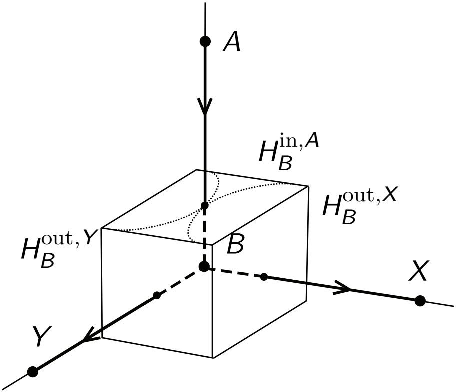
Kirk and Silber network

Subspace dynamics



Kirk and Silber network

Construction of maps

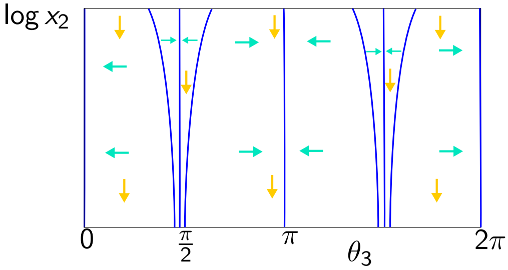


- The position the trajectory hits $H_B^{\mathrm{in},A}$ determines which equilibrium is visited next.

Kirk and Silber network

Construction of maps

- Plot the Poincare section with polar coordinates.



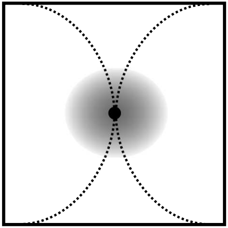
- If the network is attracting trajectories can *switch* one way but not the other.
- If network is not attracting, there can be periodic orbits lying close to either or both of the sub cycles.

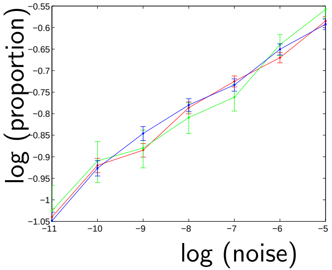
Kirk, Postlethwaite and Rucklidge, SIADS 2012

Noisy Kirk and Silber network

Armbruster, Stone and Kirk, 2003

- Consider additive noise.
- Depending on parameters, the ‘noise ellipse’ can be centered at the origin in $H_B^{\mathrm{in},A}$.
- Proportion of times each cycle visited proportional to shaded area.

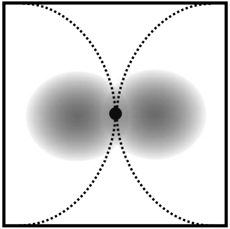




Noisy Kirk and Silber network

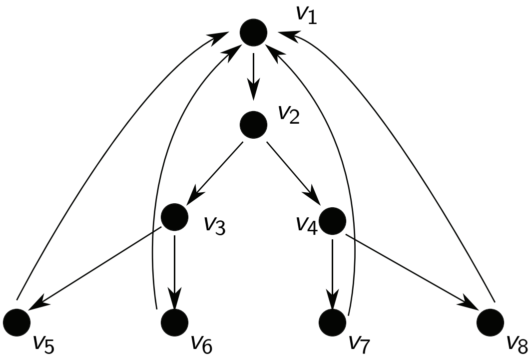
Armbruster, Stone and Kirk, 2003

- For certain parameter sets, noise ellipse can move into basin of attraction of one cycle or the other.
- This is termed *lift-off*.
- Lift-off can *reduce* switching.



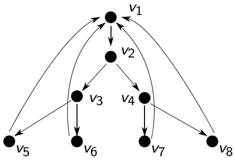
Decision network

Ashwin and Postlethwaite, 2013

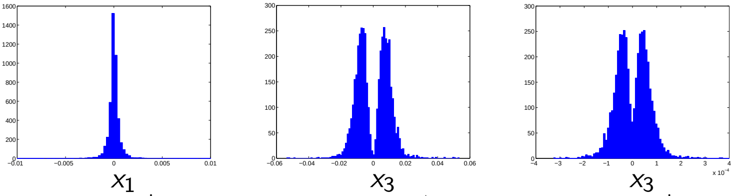


- Four sub-cycles, each with four equilibria.
- Deterministic case is very complicated: numerics show switching between sub cycles.
- Addition of noise can allow *memory*.

Decision network with memory



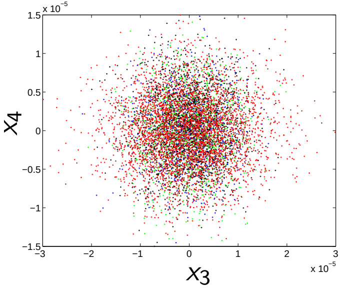
- We can choose parameters to have lift-off occur in the x_3 direction as the trajectory passes ξ_5.



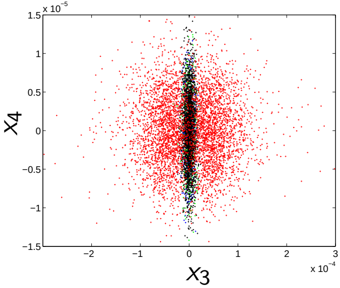
(a) on H_5^{in} (b) on H_5^{out} (c) on H_2^{in}

Decision network with memory

Without memory:



With memory:



- Red: trajectories which visited ξ_5 on the previous loop.
- Black, blue, green: trajectories which visited ξ_6, ξ_7 and ξ_8.
- Can see lift-off in the x_3 direction for red points.

Transition matrices

Without memory:

$$\begin{pmatrix} 0.46 & 0.26 & 0.18 & 0.10 \\ 0.47 & 0.23 & 0.20 & 0.10 \\ 0.49 & 0.23 & 0.18 & 0.10 \\ 0.50 & 0.22 & 0.18 & 0.10 \end{pmatrix}$$

With memory:

$$\begin{pmatrix} 0.63 & 0.33 & 0.03 & 0.02 \\ 0.49 & 0.27 & 0.16 & 0.08 \\ 0.54 & 0.24 & 0.13 & 0.09 \\ 0.52 & 0.27 & 0.14 & 0.07 \end{pmatrix}$$

Outline

Summary

- Heteroclinic cycles and networks can lose stability and produce nearby long-period periodic orbits in resonance bifurcations.
- Noisy heteroclinic cycles look like periodic orbits.
- Noisy heteroclinic networks can have much more complicated behaviour, including switching and memory.

Acknowledgements

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