\[(\Omega, X, P)\] sample space \[X: \Omega \rightarrow \mathbb{R}\] probability \[\{X \leq u\} = \{\omega \in \Omega : X(\omega) \leq u\}, \quad \omega \in \Omega\] (sample point).

Let \(M_n = \max \{X_1, X_2, \ldots, X_n\}\).

\[\exists i: i \text{ a sequence of IID RVs.}\]
\[\{M_n \leq u\} \Rightarrow \exists \omega \in \Omega: X_1(\omega) \leq u, X_2(\omega) \leq u, \ldots, X_n(\omega) \leq u.\]

\(X_i \text{ IID } \Rightarrow\)
\[P\{M_n \leq u\} = P(X_1 \leq u)P(X_2 \leq u)\ldots P(X_n \leq u)\]
\[= P(X \leq u)^n.\]

This leads to:

**Proposition:** Suppose \(\exists i: i \text{ are IID, and suppose } \exists a_n, b_n\) and a smooth function \(\tau: \mathbb{R} \rightarrow \mathbb{R}\) such that \(nP(X > \frac{u_i}{a_n} + b_n) \rightarrow \tau(u)\) as \(n \rightarrow \infty\).

Then \(\lim_{n \rightarrow \infty} P(M_n \leq a_n) = e^{-\tau(u)}\).

\[\begin{align*}
X_i &\sim \text{Exp}(1); \quad \mathbb{E}_X(x) = e^{-x} \\
R(X) &= \left(1 - e^{-\frac{u_i}{a_n} e^{-b_n}}\right) \\
\text{If } a_n &= 1, b_n = \log n. \text{ Then } P(M_n \leq u_n) \rightarrow e^{-\tau(u)}.
\end{align*}\]

\[\tau(u) = e^{-u}.\] (Gumbel)
Extremes for dynamical systems

\[(M, \mathcal{F}, \mu, \phi) \quad M: \text{Manifold} \subseteq \mathbb{R}^d\]

\[f: M \to M \text{ discrete map, } x_0 \in M, \ x_n = f^n(x_0)\]

\[\mu: \text{ ergodic, invariant measure,} \quad \mu(f^{-1}(A)) = \mu(A), \text{ and } \mu(f^{-1}(A)) \in \{0, 1\}, \]

In particular \(\forall A \subseteq M, \quad \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_A(f^k(x)) \to \mu(A)\).

\[\phi: M \to \mathbb{R} \text{ observable (cost function).}\]

Stochastic processes: \(X_n = \phi(f^{-n}x)\).

So \(\mathbb{E}[X_n] \geq 0\) if \(\mathbb{E}[X_n] = \mathbb{E}[\phi(f^{-1}x)] \geq 0\).

Invariance law: \(\mu \ni \mathbb{E}[X_n] = \mathbb{E}[\phi(f^{-n}x)] \geq 0\)

(Markov property)

Questions:
- What is the frequency of visits to \(A_n\)?
- If we visit \(A_n\), when do we visit \(A_n\) again?
- Do we visit \(A_n\) infinitely often?
- What happens if we shrink \(A_n\)?
- (Replace \(u\) by some sequence \(u_n\) such that \(\mu(A_n) \to 0\))

Remarks: If \(\sum \mu(A_n) < \infty \Rightarrow P(\limsup A_n) = 0\)

\(P(\limsup A_n = \emptyset) = 0\) by BC1.

If \(\sum \mu(A_n) = \infty \) then we need "independence" to ensure \(P(\limsup A_n) = 0\) using BC2.

However, this result turns out to be true for reasonable sets.
Suppose we can find $a_n, b_n$ and $\tau(u)$ such that
\[ n \mu \mathbb{E} x : \phi(x) \geq \frac{a}{a_n} \rightarrow \tau(u). \]
Is it true that $\mu \lbrace M_n \leq u_n \rbrace \rightarrow e^{-\tau(u)}$
\[ \mathbb{E} \mu \mathbb{E} x \in M \max \lbrace \phi(x), \phi(t_x), \ldots, \phi(f^{n-1}(x)) \rbrace \leq a_n \]
\[ \rightarrow e^{-\tau(u)}? \]

**Answer:** Yes, for reasonable $(f, M, \mu, \phi)$.

**Question:** Can we compute functional form of $\tau(u)$?

What do we need to check?

**[H]**

Merry: $f$ is mixing if for all measurable sets $A, B :$
\[ \mu(A \cap f^{-n}B) \rightarrow \mu(A) \mu(B). \]
\[ \Rightarrow \mu(x : x \in A, f^n x \in B) \rightarrow \mu(A) \mu(B). \]
\[ \Rightarrow \mu(x : f^n x \in B \mid x \in A) \rightarrow \mu(B). \]

To get "rate" of mixing we need to restrict classes of sets — or indeed look at correlation decay:
\[ \omega(n) = \left| \int \phi \cdot \mu_0 \Delta f \int \phi \Delta f + \int \phi \Delta f \right|, \]
for $\phi, \chi \in M$ is Lipschitz continuous.

$(f, \mu, M)$ has "good" mixing rate if $\exists (\theta, \Theta, \phi, \Omega > 0$
\[ \omega(n) \leq \frac{C}{n^\theta}, \quad C = C(\phi, \Omega). \]
**Examples**

- \((f, [0,1], \text{Leb})\)
  \[ f(x) = 2x \mod 1 \]
  \[ \Theta(n) \leq C r^n \quad \text{for some } r < 1. \]

- \((f, [0,1], \text{Leb})\)
  \[ f(x) = x + \alpha \mod 1 \quad \alpha \neq 0 \]
  \[ \Theta(n) \to 0 \]
  \[ \text{Not mixing.} \]

- \((f, [0,1], \mu)\)
  \[ f(x) = \begin{cases} x(1 + 2^q x^n), & x \leq \frac{1}{2} \\ 2x - 1, & x > \frac{1}{2} \end{cases} \]
  \[ \text{Tangency.} \]
  \[ \text{Here: } \frac{d\mu}{dx} \approx \frac{C}{x^q}, \quad \Theta(n) \approx n^{1/q - 1}. \quad (\text{so need } q < 1). \]

- \((f, \mathbb{R}^2, \mu)\)
  \[ f(x,y) = \left( 1 - a \frac{x+y}{h} \right) \]
  \[ \Theta(n) \leq C r^n, \quad \text{some } r < 1. \]
  \[ \mu \in \text{SRB}. \]
  \[ \mu|\mathcal{W}^u \text{ is absolutely continuous}. \]
  \[ \text{Leb}(\Lambda) = 0, \quad \Lambda = \bigcap_{n=0}^{\infty} f^n([0,1]^2). \]
  \[ \mathcal{W}^u \text{ (local manifold)}. \]
[H2] Regularity of $\mu$:

For an expanding system (non-uniformly expanding), we need
$$\frac{d\mu}{dx} = \rho(x) \in L^p$$
for some $p > 1$.

If $\mu$ is SRB, we need the local dimension to exist $\rho: a \cdot e \in M$,
$$\lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} = d.$$ (For $d = d(\mu)$ but usually independent of $x$)

[H3] Recurrence control:

Given $g: N \to \mathbb{R}$ let $E_n = \{x: d(x, f^j x) \leq \frac{1}{n}\}$, for some $j \leq g(n)^3$.

We don't want close returns too soon (typically),
so give $B \subset (0,1)$ let $g(n) = nB$. Then $\exists \tilde{B} \subset (0,1)$ such that
$$\mu(E_n) \leq \frac{1}{nB}.$$ 
[In practice $B$ close to 1 gives $\tilde{B}$ close to 0 in practice.]

Theorem (HNT, 2010). Suppose (H1)-(H3) hold for sufficiently

large $p$, $\sigma$. Suppose $\phi(x) = \phi\left(\text{dist}(x, x')\right)$

for $x \in M$ and $\phi: M^+ \to H^1$.

Then for $\mu$-a.e $x \in M$ and $\mu\{\phi(x) \leq u_n\} \to \gamma(n)$
$$\Rightarrow \mu\{M_n \leq u_n\} \to e^{-\gamma(n)}.$$

[Stated for non-uniform expanding system.]

PROBLEMS:

- Checking (H3)
- Finding $\gamma(n)$, especially if $\mu$ is SRB and $\exists \phi(x) = u_3$ has complicated geometry.
For SRB we also need condition (H2b):
\[ \exists \delta > 0, \forall r < 1, \forall y \in \mathbb{R} \]
\[ |\mu(B(x, r+y)) - \mu(B(x, r))| \leq \delta y^2 \]
\[ B(x, r) \text{: Ball of radius } r. \]

A version of \( \mu \) is in progress for the SRB case.

- Geometry:
\[ A_u = \exists \phi(x, z). \]

To get \( \gamma(u) \), we need control on \( \gamma(\mu(u)) \).

\( \mu(\mu(u)) \) easy to compute if (attractor) \( A \) has trivial geometry
and/or \( \exists \phi(x, z \in \mathbb{R}) \) are conformal to balls.

Problems if \( A \) is fractal and level regions \( \exists \phi(x, z \in \mathbb{R}) \) are not conformal to balls.

\[ \hat{\omega} \colon \phi(\hat{x}) \text{ takes max value.} \]

level set \( \exists \phi = a \).