Stability index for chaotically driven concave maps

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Overview

- Skew-product systems
- Example: baker’s map
- Invariant graph
- Inverse of baker’s map and invariant graph
- Stability index
Skew product system

- Consists of base map and fibre map
- We consider the product space $\Theta \times I$

$$F : \Theta \times I \rightarrow \Theta \times I,$$

and the skew-product system is given by

$$F(\theta, x) = (S(\theta), \hat{g}(\theta)h(x)).$$  \hfill (1)

- The first component of equation (1) is independent of $x$.
- Base map: $S\theta$
- Fibre map: $\hat{g}(\theta)h(x)$
Let $\Theta = [0, 1)^2$ and $S : \Theta \to \Theta$ be a baker's transformation

$$S(u, v) = \begin{cases} 
(s^{-1}u, sv) & \text{if } u < s \\
((1 - s)^{-1}(u - s), s + (1 - s)v) & \text{if } u \geq s 
\end{cases}$$

NB $S(u, v)$ is invertible.
The fibre maps from an interval \( I := [0, a] \) into itself of the form:

\[ x \mapsto \hat{g}(\theta)h(x) \]

where \( h(x) = \arctan(x) \).

- \( h(x) \) is invertible, strictly positive, concave and monotonic decreasing.
Invariant graph ($\hat{\phi}_\infty$)

- General meaning by Keller: is the graph of a measurable function from the base space to the fibre space which is invariant as a subset of the product space under the skew-product dynamics.
- There is a function $\hat{\phi}_\infty : \Theta \rightarrow I$ which is the invariant graph.
- The global attractor

$$K = \{(\theta, x) : 0 \leq x \leq \hat{\phi}_\infty(\theta)\}$$

- $\hat{\phi}_\infty$ is upper semicontinuous.
- $0$ is lower semicontinuous.
For a skew product system, it is well known that there exists an invariant graph from base space to fibre space whenever the contraction is uniform.

Stark (1997) has studied the invariant graphs for forced system while Glendinning (2002) studied such the graphs in a pinched skew product.

Stark proved the existence of a continuous invariant graph for the skew product.

Glendinning proved that the global attractor $K$ lies between an upper semicontinuous curve and a lower semicontinuous curve.
Mapping of invariant graph with baker’s map

\[
F \left( \begin{array}{c}
u \\
v \\
x \end{array} \right) \rightarrow \left( \begin{array}{c} S \\
g(v)h(x) \end{array} \right),
\]

with

\[
S = S_1(u, v) = \left( \frac{u}{s}, sv \right) \text{ if } 0 \leq u < s,
\]
\[
S = S_2(u, v) = \left( \frac{u-s}{1-s}, s + (1-s)v \right) \text{ if } s \leq u < 1,
\]
\[
g(v) = r(1 + \varepsilon + \cos(2\pi v)),
\]
\[
h(x) = \arctan(x),
\]

where \( s = 0.45, \varepsilon = 0.01, r = 2.5 \).
Figure: The attractor which is the invariant graph $\hat{\varphi}_\infty(v)$ for the baker's map $x$ vs. $v$ for $r = 2.5$. 
Figure: The three-dimensional invariant graph for the baker’s map $u, v, x$. 
Inverse of baker’s map and invariant graph

Let $G = F^{-1}$ and by ignoring the (') sign, the inverse maps are:

i) If $0 \leq v < s$, then

$$G(u, v, x) = \left( su, s^{-1}v, \tan\left(\frac{x}{g\left(\frac{v}{s}\right)}\right)\right).$$

ii) If $s \leq v < 1$, then

$$G(u, v, x) = \left( (1 - s)u + s, (1 - s)^{-1}(v - s), \tan\left(\frac{x}{g\left(\frac{v-s}{1-s}\right)}\right)\right).$$

By using this inverse map, we will plot the basin of attraction in the journey to compute the stability index.
Figure: The basin of attraction for the inverse of invariant graph and the baker’s map for $r = 2.5$. The black area denotes the basin where the points go to $x = 0$ and the orange area denotes the points go to $x = \infty$. 
Figure: $r = 2.5$. (a) The attractor invariant graph $\hat{\phi}_\infty(\nu)$ for baker’s map. (b) The basin of attraction for the inverse of invariant graph and the baker’s map. As we iterate backward, the point will either go to 0 or $\infty$. 
What is stability index?

- Introduced by Podvigina and Ashwin (2011).
- Consider a point \( x \in X \), and defined that
  \[
  \Sigma_\epsilon(x) = \frac{\ell(B_\epsilon(x) \cap N)}{\ell(B_\epsilon(x))},
  \]
  where \( B_\epsilon(x) \) is a ball of radius \( \epsilon \) about \( x \), \( N \) is basin of attraction and \( \ell(\cdot) \) denotes Lebesgue measure on \( \mathbb{R}^n \).
- Note that \( 0 \leq \Sigma_\epsilon(x) \leq 1 \) and define that \( \sigma(x) = [-\infty, \infty] \).

**Definition**

*For a point \( x \in X \), the stability index of \( X \) at \( x \) to be*

\[
\sigma(x) := \sigma_+(x) - \sigma_-(x),
\]

*which exists when the following converge:*

\[
\sigma_-(x) := \lim_{\epsilon \to 0} \frac{\ln(\Sigma_\epsilon(x))}{\ln \epsilon}, \quad \sigma_+(x) := \lim_{\epsilon \to 0} \frac{\ln(1 - \Sigma_\epsilon(x))}{\ln \epsilon}.
\]

- \( \sigma(x) \) of a point \( x \in X \) characterizes the local geometry of the basin of attraction of \( X \).
Stability index $\sigma(\nu)$

- Keller computed $\sigma(\nu)$ for the global attractor $K$ for $F$.
- He pick a point $(\nu, 0)$ on invariant graph and define a local stability index $\sigma(\nu)$ in the following way:

$$\sigma(\nu) = \sigma_+(\nu) - \sigma_-(\nu),$$

where

$$\sigma_-(\nu) = \lim_{\epsilon \to 0} \frac{\log \Sigma_\epsilon(\nu)}{\log \epsilon}, \quad \sigma_+(\nu) = \lim_{\epsilon \to 0} \frac{\log(1 - \Sigma_\epsilon(\nu))}{\log \epsilon},$$

with

$$\Sigma_\epsilon(\nu) = \frac{1}{\epsilon \vert U_\epsilon(\nu) \vert} \int_{U_\epsilon(\nu)} \min\{\varphi_\infty(t), \epsilon\} dt$$

and

$$1 - \Sigma_\epsilon(\nu) = \frac{1}{\epsilon \vert U_\epsilon(\nu) \vert} \int_{U_\epsilon(\nu)} (\epsilon - \varphi_\infty(t))^+ dt.$$

$U_\epsilon(\nu) = (\nu - \epsilon, \nu + \epsilon)$ with size $2\epsilon$-symmetric interval nbhd.

- In our case, we aim to compute $\sigma(\nu)$ for the basin of attraction 0 for $F^{-1}$. 
This is computing using $F^{-1}$.

Figure: The basin of attraction by using random number generator. The blue dots denote the basin where the points go to $x = 0$ and the yellow dots denote the points go to $x = \infty$. 

(a) $r = 1.74$  
(b) $r = 2.0$  
(c) $r = 2.5$
Figure: The proportion of the blue dots over the whole image for $r = 1, \ldots, 5$. We can see that the proportion is increase as we increase the parameter $r$. 
Figure: The proportion of the blue dots over the $\epsilon$ for $r = 2.5$. The proportion decrease with the increase of $\epsilon$. The proportion 1 means that the neighbourhood only contains the blue points whereas the proportion $< 1$ shows that the blue and yellow points are mixed together.
Figure: Computation of $\sigma_+(v)$: $\log(1 - \Sigma_\epsilon(v))$ vs. $\log(\epsilon)$ for $r = 2.5$. Here the slope is $\infty$. 
Figure: Computation of $\sigma_-(v)$: $\log(\Sigma_\epsilon(v))$ vs. $\log(\epsilon)$ for $r = 2.5$. Here the slope is 0.
Figure: Stability index for Keller’s paper for $r = 1, \ldots, 3$. Here the index ranges from $-\infty$ to $\infty$ where in this figure $-1$ represents the $-\infty$ and $1$ represents $\infty$. Therefore $\sigma(v) = [-\infty, \infty]$. 
Future work

- Generalize stability index rather than just for a point.
- Use stability index to characterize the invariant graphs in terms of Lyapunov exponents in the case of riddled basins.
References


THANKS FOR LISTENING!