Dynamics Reading Group
Most likely paths

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Consider a deterministic ODE:

\[ \dot{x} = \mu(x(t)) \]

For a given starting position, end point and time there is one solution.

But if we were to add noise to this then suddenly there are many different paths possible and we’re interested in finding the most likely path.
Introduction

Objectives:

- Calculate the most likely path from a known starting position to a known end position in a given time.
- What is the probability of following this path?
- Is there an optimum time to make the transition?

Brownian motion of a particle is governed by the stochastic differential equation:

$$dX_t = \mu(X_t, t)dt + \sqrt{2D(X_t, t)}dW_t$$

with drift $\mu(X_t, t)$ and $D(X_t, t)$ is a diffusion coefficient.
Introduction

- Probability density function of the random variable $X_t$ is governed by the Fokker-Planck equation:

$$
\frac{\partial P(x, t)}{\partial t} = -\frac{\partial}{\partial x}(\mu(x, t)P(x, t)) + \frac{\partial^2}{\partial x^2}(D(x, t)P(x, t))
$$

where a potential well $U(x, t)$ satisfies:

$$
\frac{dU(x, t)}{dx} = -\mu(x, t)
$$
Figure: Fokker-Planck run for $\dot{x} = -1 + \eta$, $x_0 = 0$, $T = 3$
Case 1: Pure diffusion

\[ \tilde{P} \propto \exp \left( - \frac{1}{4D} \int_{t_0}^{T} \left( \frac{dx}{d\tau} \right)^2 d\tau \right) \]
Key book results

- Case 1: Pure diffusion

\[ \tilde{P} \propto \exp \left( - \frac{1}{4D} \int_{t_0}^{T} \left( \frac{dx}{d\tau} \right)^2 d\tau \right) \]

- Case 2: annihilation model

\[ \tilde{P} \propto \exp \left( - \frac{1}{4D} \int_{t_0}^{T} \left( \frac{dx}{d\tau} \right)^2 d\tau - \int_{t_0}^{T} A(x(\tau), \tau) d\tau \right) \]
Case 3: Most likely path will maximise:

\[ W(x(\tau)) = \exp\left(\frac{U(x_0) - U(x_T)}{2D}\right) \exp\left(-\int_{t_0}^{T} \mathbb{L}(x(\tau))\,d\tau\right) \]

where

\[ \mathbb{L} = \frac{1}{4D} \left(\frac{dx}{d\tau}\right)^2 + V_s \]

and

\[ V_s = \frac{1}{4D} \left(\frac{dU}{dx}\right)^2 - \frac{1}{2} \frac{d^2U}{dx^2} \]
In order to minimise need to solve the Euler-Lagrange equation:

\[
\frac{\partial L}{\partial x} - \frac{d}{d\tau} \frac{\partial L}{\partial \dot{x}} = 0
\]

From this a 2\textsuperscript{nd} order ODE is created that the most likely trajectory will satisfy:

\[
\ddot{x} = 2D \frac{dV_s}{dx}
\]
Example

Consider the example:

\[ \frac{dx_t}{dt} = -ax_t dt + \sigma dW_t \]

Most likely path satisfies:

\[ \ddot{x} - a^2 x = 0 \]

In this case the solution can be obtained analytically:

\[ x(t) = x_0 e^{at} + (x_T - x_0 e^{aT}) \frac{\sinh(at)}{\sinh(aT)} \]
Figure: Most likely path for constant single potential well, \( a = 1, x_0 = 10, x_T = 0, T = 10 \)
Time dependent potential wells

- Time dependent potential wells of the form $U(x, t)$
- New PDE is:
  \[ \frac{\partial P_s}{\partial t} = D \frac{\partial^2 P_s}{\partial x^2} + \left( \frac{U''}{2} - \frac{U'^2}{4D} + \frac{\dot{U}}{2D} \right) P_s \]
- $2^{nd}$ order ODE remains the same with just an extra term in $V_s$:
  \[ \ddot{x} = 2D \frac{dV_s}{dx} \]
Example

- Suppose now $a$ is not constant, $a(t) = a_0 - \epsilon t$, with drift speed $\epsilon$
- The most likely path now satisfies:
  \[ \ddot{x} = (a(t)^2 + \epsilon)x \]
- To be solved numerically
Figure: Most likely path for time dependent single potential well, $a_0 = 0.05$, $\epsilon = -0.07$, $x_0 = 10$, $x_T = 0$, $T = 10$
**Example**

*Figure*: Optimal time for constant single potential well, \( a = 0.4, D = 0.01, x_0 = 2 \)
