Bifurcation analysis for systems with rotational symmetry & delay

$$\dot{x}(t) = f(x(t), x(t-\tau), p) \qquad x(t) \in \mathbb{R}^n, p \in \mathbb{R}^k, \tau \geq 0$$

, r.h.s. f satisfies

$$e^{A\vartheta} f(x, y, p) = f(e^{A\vartheta} x, e^{A\vartheta} y, p) \quad \text{where}$$

$$A \in \mathbb{R}^{n \times n}, \quad e^{A \cdot 2\pi} = I, \quad A^T = -A$$

Relative Equilibria: solutions of type $x(t) = e^{A\omega t} x_0$

Relative Periodic Orbits: solution $x(t)$ s.t. there ex. $T > 0, \vartheta \in [0, 2\pi)$ and

$$x(t+T) = e^{A\vartheta} x(t) \quad \text{for all } t \in \mathbb{R}$$

Example: Lang-Kobayashi-Eq.

$$\dot{E}(t) = (1+\alpha) n(t) E(t) + \eta e^{i\varphi} E(t-\tau)$$
$$\dot{n}(t) = \varepsilon \cdot (I - n(t) - (2n(t)+1) \overline{E(t)}^T E(t))$$

$E(t) \in \mathbb{C} = \mathbb{R}^2$, $n(t) \in \mathbb{R}$, $A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Basic Eq.: $E = 0, \; n = I$

RE: $E(t) = E_0 e^{i\omega t}$ $(|E(t)| = \text{const})$, $n = n_0$

Basic bifurcation analysis (I) REs

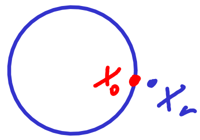
Defining system for RE: Unknowns: $x_0 \in \mathbb{R}^n, \omega \in \mathbb{R}$ # $n+1$

$$\boxed{0 = A\omega x_0 - f(x_0, e^{-A\omega\tau} x_0, p)} \qquad n \text{ Eqs.}$$

need one add. Eq. to determine ω

If (x_0,ω) is RE then $(e^{A\vartheta}x_0,\omega)$ is also RE for all $\vartheta\in[0,2\pi)$
Assume we have reference x_r. Pick x_0 s.t.

$$\|e^{A\vartheta}x_0-x_r\|^2 \text{ is minimal at } \vartheta=0$$



$$\hookrightarrow \quad \frac{\partial}{\partial\vartheta}\left[e^{A\vartheta}x_0-x_r\right]^T\left[e^{A\vartheta}x_0-x_r\right]\Big|_{\vartheta=0}=0$$

$$(Ax_0)^T[x_0-x_r]+[x_0-x_r]^T(Ax_0)=0 \qquad (x_0^TAx_0=0)$$

$$\hookrightarrow \quad \boxed{x_r^TAx_0=0} \quad \leftarrow \text{add. Eq.}$$

When tracking RE in $p\in\mathbb{R}^1$ x_r = prev. sol. x_0 along curve
$n+1$ Eqs. $n+1+k$ Vars $k=1\to$ curve

Defining system for RPO: periodic BVP on interval $[0,1]$

unknowns: $x(t)\in C_p^1([0,1],\mathbb{R}^n)$, $T\in\mathbb{R}^1$, $\omega\in[0,2\pi)$, $p\in\mathbb{R}^k$
in rotating coordinates:
$x_{old}(t)=e^{A\omega t}x_{new}(t)$, $\dot{x}(t)=-A\omega x(t)+f(x(t),e^{-A\omega\tau}x(t-\tau),p)$
new x has period $T\to$ rescale time by T:

$$\boxed{\begin{aligned}\dot{x}(t)&=-AT\omega x(t)+Tf\left(x(t),e^{-A\omega\tau}x\left(t-\tfrac{\tau}{T}\right),p\right)\\ x(0)-x(1)&=0 \qquad n\ \text{BC} \qquad \text{mod } [0,1]\end{aligned}}$$

periodic BVP

need 2 more Eqs to determine T,ω
If $x(t)$ solves BVP then $e^{A\vartheta}x(t+s)$ solves BVP for all $\vartheta\in[0,2\pi)$, $s\in[0,1]$
2par $\nearrow$ family.
Assume $x_r\in C_p^1([0,1];\mathbb{R}^n)$ given, choose $x(t)$ s.t.

$$\int_0^1\|e^{A\vartheta}x(t+s)-x_r(t)\|^2\,dt \quad \text{is minimal at } s=0,\ \vartheta=0$$

$$\hookrightarrow \frac{\partial}{\partial\vartheta}\Big|_{\vartheta=0,s=0}=0: \qquad \int_0^1 x_r(t)^TAx(t)\,dt=0$$

$$\frac{\partial}{\partial s}\Big|_{\vartheta=0,s=0}=0: \qquad \int_0^1 x_r'(t)^Tx(t)\,dt=0$$

2 eqs.

BVP with 2 add. Eqs for add unknowns $\omega, T, p \in \mathbb{R}^1$ → Curve

x_r : previous point on curve

Bifurcations of RE: Fold

$$\left.\begin{aligned} 0 &= A\omega x_0 - f(x_0, e^{-A\omega\tau} x_0, p) \\ 0 &= x_r^T A x_0 \end{aligned}\right\} 0 = F(X,p),\ X = (x_0, \omega) \qquad 2n+2 \text{ eqs}$$

$$0 = \partial_X F(X,p) Y \qquad Y \in \mathbb{R}^{n+1}$$

$$0 = Y^T Y - 1$$

Vars: $x_0 \in \mathbb{R}^n, \omega \in \mathbb{R}, Y \in \mathbb{R}^{n+1}, p \in \mathbb{R}^k$, $k=2$ → Curve

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Hopf:

	Eqs
$0 = A\omega x_0 - f(x_0, e^{-A\omega\tau} x_0, p)$	n
$0 = x_r^T A x_0$	1
$0 = A\omega y_0 - \partial_1 f(\ ) y_0 - \partial_2 f(\ ) e^{-A\omega\tau} e^{-2\pi i \tau/T} y_0 - \frac{2\pi i}{T} y_0$	$2n$
$0 = \bar{y}_0^T y_0 - 1$	1
$0 = \mathrm{Re}\, y_0^T \mathrm{Im}\, y_0$	1
	$3n+3$

Vars: $x_0 \in \mathbb{R}^n$, $\omega, T \in \mathbb{R}^1$, $y_0 \in \mathbb{C}^n = \mathbb{R}^{2n}$, $p \in \mathbb{R}^k$, $k=2$ → Curve

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Bifurcations of RPOs: Fold, Period Doubling, Torus bif.

Fold:

$$\left.\begin{aligned} 0 &= \dot{x}(t) + AT\omega x(t) - T f\Big(x(t), e^{-A\omega\tau} x\big(\overbrace{t - \tfrac{\tau}{T}}^{\text{mod } [0,1]}\big), p\Big) \\ 0 &= x(0) - x(1) \\ 0 &= \int_0^1 x_r(t)^T A x(t)\, dt \\ 0 &= \int_0^1 x_r'(t)^T x(t)\, dt \end{aligned}\right\} \begin{aligned} &0 = F(X,p) \\ &X = (x(\cdot), T, \omega) \end{aligned}$$

$$0 = \partial_X F(X,p) Y \qquad Y = (\hat{x}(\cdot), \hat{T}, \hat{\omega})$$

$$0 = \int_0^1 \hat{x}(t)^T \hat{x}(t)\, dt + \hat{T}^2 + \hat{\omega}^2 - 1$$

$2n$ diffeqs for $x(\cdot)$, $\hat{x}(\cdot)$, 5 conditions for $T, \omega, \hat{T}, \hat{\omega}, p$ $\quad \dim_p = 2$

$\left[\frac{\partial F}{\partial T}\hat{T} \text{ involves } \quad -T\partial_2 f(\;)\,\dot{x}(t-\tfrac{\tau}{T})\,\tfrac{\tau}{T^2}\,\hat{T} \quad \hookrightarrow \;\dots\; x(t-\tfrac{2\tau}{T})\right]$

Torus Bif & Period doubling

Floquet problem: linearization of DDE w.r.t. x

$$\frac{d}{dt}\hat{x}(t) = -AT\omega\hat{x}(t) + T\partial_1 f(\;)\hat{x}(t) + T\partial_2 f(\;)e^{-\omega\tau}\hat{x}(t-\tau)$$

look for sol. satisfying $\hat{x}(1) = e^{i\vartheta}\hat{x}(0)$

$\hat{x}(t) = e^{i\vartheta t}y(t)$

$$\boxed{\begin{aligned} i\vartheta y(t) + \dot{y}(t) &= -AT\omega y(t) + T\partial_1 f\, y(t) + T\partial_2 f\, e^{-\omega\tau - i\vartheta\tau} y(t-\tau) \\ 0 &= y(1) - y(0) \end{aligned}}$$

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$$\left[\begin{aligned} 0 &= \dot{x}(t) + AT\omega x(t) - Tf(x(t), e^{-A\omega\tau}x(t-\tfrac{\tau}{T}), p) \\ 0 &= x(0) - x(1) \\ 0 &= \int_0^1 x_r(t)^T A x(t)\, dt \\ 0 &= \int_0^1 x_r'(t)^T x(t)\, dt \end{aligned}\right]$$

$y(t) \in \mathbb{C}^{2n} = \mathbb{R}^{2n}$, $\vartheta \in (0, \pi]$

$0 = \int_0^1 \bar{y}^T(t)\, y(t)\, dt - 1$

$0 = \int_0^1 \operatorname{Re} y(t)^T \operatorname{Im} y(t)\, dt$

$3n$ diff. eqs. for $x(\cdot)$, $\operatorname{Re} y(\cdot)$, $\operatorname{Im} y(\cdot)$, 4 conditions for $\omega, T, \vartheta, p \rightarrow \dim p = 2$

Special case: $\vartheta = \pi \rightsquigarrow$ Period doubling

Hopf, Period doubling, Torus bif are identical to generic case except add. variable ω & condition $0 = \langle x_r, Ax \rangle$