Closed orbits in quotient systems

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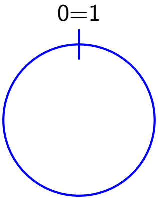
17 February 2015

Closed Orbits

Let $X = \mathbb{R}/\mathbb{Z}$, and let $T : X \to X$ be the circle doubling map defined by $T(x) = 2x \pmod 1$.



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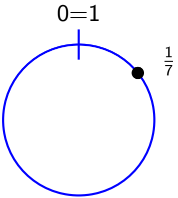

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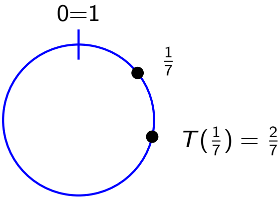

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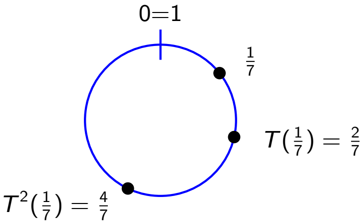

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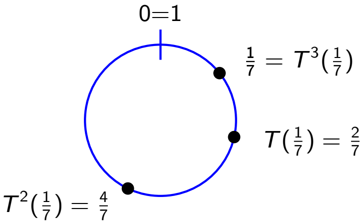

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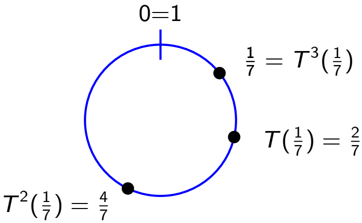


In general, for n a natural number, we denote by

$$F_n(T) = \#\{x \in X : T^n(x) = x\}$$

the number of points of period n under T, and by

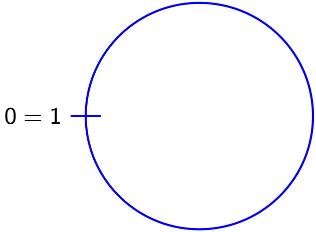
$$O_n(T) = \#\{\mathfrak{O}_T(x) : |\mathfrak{O}_T(x)| = n\} = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) F_d(T),$$

we denote the number of closed orbits of length n under T.

Quotient Systems

Let $X = \mathbb{R}/\mathbb{Z}$, and let $T : X \to X$ be the circle doubling map. Suppose C_2 is the finite cyclic group of two elements $\{1, g\}$ acting on X, where we define the action of C_2 by $g(x) = 1 - x$.

X



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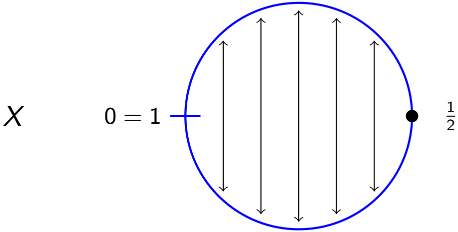

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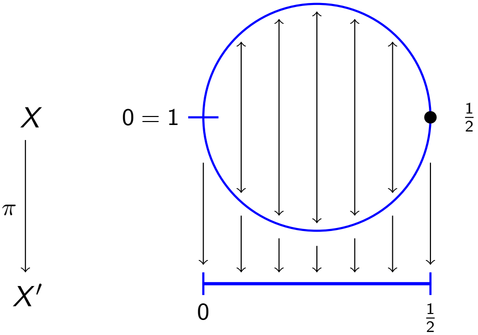
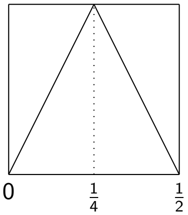


We let $\pi(x) = [x]$, for all $x \in X$, where $[x] = \{x, 1 - x\}$ is the equivalence class of x under the action of C_2.

The induced map $T' : X' \to X'$ defined on the quotient space $X' = [0, \frac{1}{2}]$ is given by the tent map:

$$T'(x) = \begin{cases} 2x, & \text{if } \ 0 \le x \le \frac{1}{4}; \\ 1 - 2x, & \text{if } \ \frac{1}{4} \le x \le \frac{1}{2}. \end{cases}$$



We have the following number of periodic points for the circle doubling map T and the tent map T', respectively:

$$F_n(T) = 2^n - 1 \qquad \text{and} \qquad F_n(T') = 2^n,$$

for all $n \ge 1$. Then

$$F_n(T) \sim F_n(T') \quad \text{as } n \to \infty,$$

showing the same asymptotic growth rate for periodic points (and therefore orbits) of the circle doubling map T and the tent map T'.

Quotient Systems

Let (X, d) be a compact metric space and $T : X \to X$ be continuous. Let G be a finite group acting on X where the action commutes with T.

$$X \xrightarrow{\quad T \quad} X$$

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$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "\pi"'] & X \arrow[d] \\ X' & X' \end{tikzcd} \qquad \pi(x) = \mathfrak{O}_G(x) = \{g(x) : g \in G\}$$

We define

$$X' = G \backslash X = \{\mathfrak{O}_G(x) : x \in X\}$$

to be the set of all orbits under the action of G, and we note that (X', d') is a compact metric space.

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We define

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to be the set of all orbits under the action of G, and we note that (X', d') is a compact metric space. Further, we have that the induced map $T' : X' \to X'$ on the quotient space is defined by

$$T'(\mathfrak{O}_G(x)) = \mathfrak{O}_G(T(x)).$$

Then $\pi(T(x)) = T'(\pi(x))$, and we have that π is a topological semi-conjugacy.

Note that for two topologically semi-conjugate maps T and T' there is, in general, no relationship between the count of periodic points (and closed orbits) of T and T'.

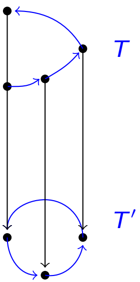

Figure 1: Creating closed orbits from non-closed orbits

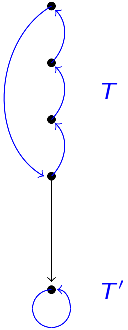

Figure 2: Squashing closed orbits to a single closed orbit of length 1 (a fixed point)

Glueing, Shortening, and Surviving Orbits

We have that three phenomena occur in quotient systems: surviving orbits, glueing orbits, and shortening orbits. Note that it is possible for the two phenomena, glueing and shortening, to occur at the same time.

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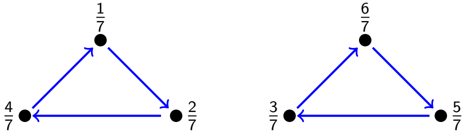

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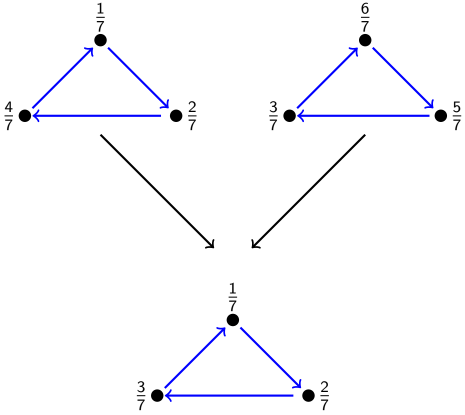

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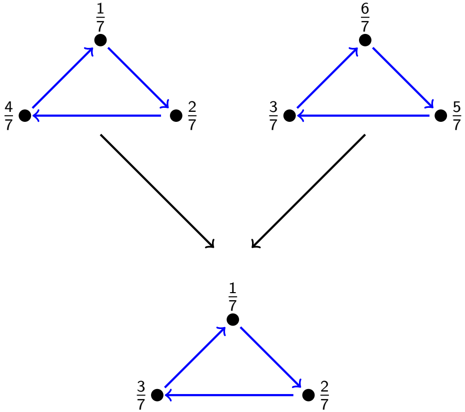
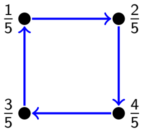


shortening orbit



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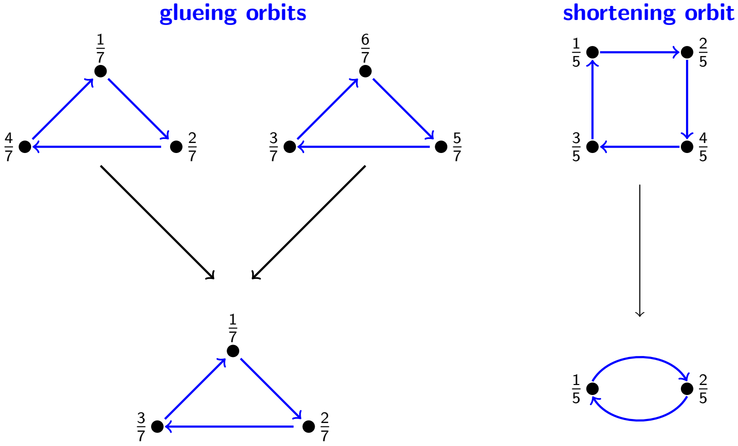


Glueing, Shortening, and Surviving Orbits

Lemma 1

Let $T : X \to X$ be a map defined on a set X. Let G be a finite group acting on X where the action of G commutes with T and let $k \in \mathbb{N}$. Then the orbit $\mathfrak{O}_T(x)$ of $x \in X$ shortens in length by a factor of $\frac{1}{k}$ if and only if $|\mathfrak{O}_T(x) \cap \mathfrak{O}_G(x)| = k$. Moreover, if $\mathfrak{O}_T(x)$ shortens in length by a factor of $\frac{1}{k}$, then there exists an element $g \in G$ such that $\mathfrak{O}_T(x) \cap \mathfrak{O}_G(x) = \{g^i(x) : i \in \mathbb{N}_0\} = \mathfrak{O}_{\langle g \rangle}(x)$.

Lemma 2

Let $T : X \to X$ be a map defined on a set X. Let G be a finite group acting on X where the action of G commutes with T, and let $x \in X$. Then the number of orbits that glue to $\mathfrak{O}_T(x)$ (including itself) is given by

$$\frac{|\mathfrak{O}_G(x)|}{|\mathfrak{O}_T(x) \cap \mathfrak{O}_G(x)|}.$$

We define

$$G_x = \{g \in G : g(x) = x\}$$

to be the **stabilizer** of x, and

$$N_G(H) = \{g \in G : gHg^{-1} = H\}$$

to be the **normalizer** of a subgroup $H \leq G$. Then for $H \in P(G)$, we write

$$[H] = \{gHg^{-1} : g \in G\}$$

for its conjugacy class in $\overline{P}(G)$. We define

$$X_H = \{x \in X : G_x = H\};$$
$$X_{[H]} = \{x \in X : G_x \in [H]\}.$$

Now, let $\mathbb{X}$ be the **periodic set**, that is $\mathbb{X}$ is the subset of all periodic points of X under T. Then T preserves each set $\mathbb{X}_{[H]} = X_{[H]} \cap \mathbb{X}$ and T' preserves each set $\mathbb{X}'_{[H]} = G \backslash \mathbb{X}_{[H]}$.

$$\mathbb{X} = \bigsqcup_{[H] \in \overline{P}(G)} \mathbb{X}_{[H]} \qquad \mathbb{X}' = \bigsqcup_{[H] \in \overline{P}(G)} \mathbb{X}'_{[H]}.$$

For any group H, set

$$\Delta(H) = \{|\langle h\rangle| : h \in H\}$$

to be the set of all orders of elements of H.

Lemma 3

Given the orbit $\mathfrak{O}_T(x)$ of $x \in X_{[H]}$, there exists $k \in \Delta\left(N_G(H)/H\right)$ such that

(i) *$\mathfrak{O}_T(x)$ shortens in length by a factor of $\frac{1}{k}$, and*

(ii) *$\mathfrak{O}_T(x)$ glues to $\frac{[G:H]}{k}$ orbits (including itself).*

Now, let $[H] \in \overline{P}(G)$, and define

$$\Sigma_{[H]} = \left\{ (\delta, \theta) : \delta \in \Delta(N_G(H)/H), \theta = \tfrac{[G:H]}{\delta} \right\},$$

Fix $I = \{1\}$ to be the trivial subgroup of G, so that

$$\Sigma_{[I]} = \left\{ (\delta, \theta) : \delta \in \Delta(G), \theta = \tfrac{|G|}{\delta} \right\}.$$

For $\sigma = (\delta_\sigma, \theta_\sigma) \in \Sigma_{[H]}$, write $O_n^{[H]_\sigma}(T)$ for the number of orbits of length n in $\mathbb{X}_{[H]}$ such that $|\mathfrak{O}_T(x) \cap \mathfrak{O}_G(x)| = \delta_\sigma$ and $\frac{[G:H]}{\delta_\sigma} = \theta_\sigma$. Then

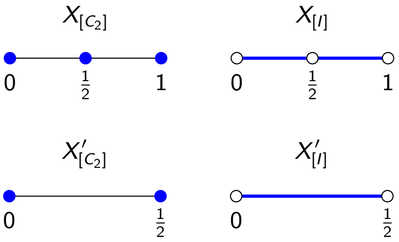
$$O_n(T) = \sum_{[H] \in \overline{P}(G)} \sum_{\sigma \in \Sigma_{[H]}} O_n^{[H]_\sigma}(T),$$
$$O_n(T') = \sum_{[H] \in \overline{P}(G)} \sum_{\sigma \in \Sigma_{[H]}} \left(\tfrac{1}{\theta_\sigma}\right) O_{\delta_\sigma n}^{[H]_\sigma}(T),$$

for all $n \geq 1$. Here, we understand that we have the following two constraints:

$(\mathscr{C}_1)$ $\theta_\sigma \mid O_n^{[H]_\sigma}(T)$, $\qquad$ $(\mathscr{C}_2)$ $O_n^{[H]_\sigma}(T) = 0$ if $\delta_\sigma \nmid n$.

Glueing, Shortening, and Surviving Orbits

$$
\begin{array}{rcr}
[C_2] = [\{1, g\}] = \{\{1, g\}\}; & X_{[C_2]} = \left\{0, \tfrac{1}{2}\right\}; & X'_{[C_2]} = \left\{0, \tfrac{1}{2}\right\}; \\
[I] = [\{1\}] = \{\{1\}\}; & X_{[I]} = \left(0, \tfrac{1}{2}\right) \cup \left(\tfrac{1}{2}, 1\right); & X'_{[I]} = \left(0, \tfrac{1}{2}\right).
\end{array}
$$



$X_{[C_2]}$ We have $N_{C_2}(C_2) = C_2$. Then $|\mathfrak{O}_{C_2}(x) \cap \mathfrak{O}_T(x)| = 1$ and $[C_2 : C_2] = 1$. It follows that

$$O_n^{[C_2]}(T') = O_n^{[C_2](1,1)}(T)\,,$$

for all $n \geq 1$.

$X_{[I]}$ We have $N_{C_2}(I) = C_2$. Then $|\mathfrak{O}_{C_2}(x) \cap \mathfrak{O}_T(x)| = k \in \{1,2\}$ and $\frac{[C_2:I]}{k} = \frac{2}{k}$. It follows that

$$O_n^{[I]}(T') = \tfrac{1}{2}\, O_n^{[I](1,2)}(T) + O_{2n}^{[I](2,1)}(T)\,,$$

for all $n \geq 1$.

Then the general relationship between the orbits of the circle doubling map T and the tent map T' is given by

$$O_n(T') = O_n^{[C_2]}(T') + O_n^{[I]}(T')\,,$$

for all $n \geq 1$.

We have the following bounds for periodic points and orbits in quotient systems:

Lemma 4

Let (X', T') be the quotient system of (X, T) under the action of a finite group G. Then, for any $n \geq 1$, we have

($\mathscr{B}_1$) $\frac{F_n(T)}{|G|} \leq F_n(T') \leq F_n(T) + \sum_{\substack{\sigma \in \Sigma_{[l]} \\ \delta_\sigma > 1}} \left(\frac{1}{\delta_\sigma \theta_\sigma}\right) F_{\delta_\sigma n}(T)$;

($\mathscr{B}_2$) $O_n(T') \leq O_n(T) + \sum_{\substack{\sigma \in \Sigma_{[l]} \\ \delta_\sigma > 1}} \left(\frac{1}{\theta_\sigma}\right) O_{\delta_\sigma n}(T)$, *and if n is such that $\delta_\sigma \nmid n$, for any $\sigma \in \Sigma_{[l]}$ such that $\delta_\sigma > 1$, then* $O_n(T') \geq \frac{O_n(T)}{|G|}$.

Now, let

$$\nabla = \max\{\delta : \delta \in \Delta(G)\}$$

be the largest order of any element in G. Denote by H_∇ a maximal subgroup of G such that $N_G(H_\nabla)/H_\nabla$ contains an element of order ∇. Then

$$\Theta = \frac{[G:H_\nabla]}{\nabla}$$

denotes the minimum value for θ_σ such that $\sigma = (\nabla, \theta_\sigma)$.

Lemma 4 gives an immediate result concerning the logarithmic growth rate in $F_n(T')$:

Corollary 5

Let (X', T') be the quotient system of (X, T) under the action of a finite group G, and suppose there exists $\eta > 0$ such that $\limsup_{n\to\infty} \frac{1}{n} \log F_n(T) = \eta$. Then

$$\eta \leq \limsup_{n\to\infty} \frac{1}{n} \log F_n(T') \leq \nabla\eta.$$

Later, in Corollary 9 we will show that any growth rate in between the bounds of Corollary 5 can be achieved.

The Super Basic Lemma

We now observe that if we are free to choose a pair of topological dynamical systems, then ($\mathscr{C}_1$) and ($\mathscr{C}_2$) are the only constraints on the orbit behaviour under a finite group action.

Proposition 6

Let G be a finite group, and let $\left(b_n^{[H]_\sigma}\right)_{n=1}^{\infty}$ be a sequence of non-negative integers, for $[H] \in \overline{P}(G)$ and $\sigma \in \Sigma_{[H]}$, such that $b_1^{[G]_{(1,1)}} \geq 1$. Define $\left(a_n^{[H]_\sigma}\right)_{n=1}^{\infty}$ by

$$a_n^{[H]_\sigma} = \begin{cases} \theta_\sigma \, b_{n/\delta_\sigma}^{[H]_\sigma}, & \text{if } \delta_\sigma \mid n, \\ 0, & \text{otherwise.} \end{cases}$$

Further, define

$$a_n = \sum_{[H]\in\overline{P}(G)} \sum_{\sigma\in\Sigma_{[H]}} a_n^{[H]_\sigma} \qquad b_n = \sum_{[H]\in\overline{P}(G)} \sum_{\sigma\in\Sigma_{[H]}} b_n^{[H]_\sigma},$$

for all $n \geq 1$. Then there exist a topological dynamical system (X, T) and an action of G on X which commutes with T such that

$$O_n(T) = a_n \qquad \text{and} \qquad O_n(T') = b_n, \quad \text{for all } n \geq 1.$$

The following theorem shows that it is possible to impose conditions directly on the sequences (a_n) and (b_n) to guarantee that the combinatorial constraints $(\mathscr{C}_1)$ and $(\mathscr{C}_2)$ are satisfied:

Theorem 7

Let G be a finite group and let (a_n) be a sequence of non-negative integers with $a_1 \geq 1$ such that there exists $N > 0$ with $\frac{a_{\nabla n}}{\Theta} \geq a_n$ for $n \geq N$. Further, let (b_n) be any sequence of non-negative integers such that $b_1 > \frac{a_1}{|G|}$ and

$$\begin{cases} \frac{a_n}{|G|} \leq b_n \leq a_n, & \text{for } n < N, \\ a_n \leq b_n \leq \frac{a_{\nabla n}}{\Theta}, & \text{for } n \geq N. \end{cases}$$

Then there exist a topological dynamical system (X, T) and an action of G on X which commutes with T such that

$$O_n(T) = a_n \qquad \text{and} \qquad O_n(T') = b_n, \qquad \text{for all } n \geq 1.$$

Corollary 8

Let (X', T') be the quotient system of (X, T) under the action of a finite group G, and suppose there exists $\eta > 0$ such that $\limsup_{n\to\infty} \frac{1}{n} \log F_n(T) = \eta$. Then

$$\eta \leq \limsup_{n\to\infty} \frac{1}{n} \log F_n(T') \leq \nabla\eta.$$

Corollary 9

Let G be a finite group. Suppose $1 < \lambda \in \mathbb{R}$, $\eta \in \mathbb{R}$ and $c \in \mathbb{R}^+$ are such that either

(i) $\eta = \lambda$ *and* $c \geq \frac{1}{|G|}$, *or*

(ii) $\lambda < \eta < \lambda^\nabla$, *or*

(iii) $\eta = \lambda^\nabla$ *and* $c \leq \frac{1}{\Theta}$.

Then there exist a topological dynamical system (X, T) and an action of G on X which commutes with $T : X \to X$, such that

$$O_n(T) \sim \lambda^n \qquad \text{and} \qquad O_n(T') \sim c\eta^n \qquad \text{as } n \to \infty.$$

The Dynamical Zeta Function

The dynamical zeta function for any map $T : X \to X$, such that $F_n(T) < \infty$ for all $n \geq 1$, is defined by

$$\zeta_T(z) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} F_n(T)\right) .$$

- Rationality of the dynamical zeta function implies nice and well behaved growth rates for the sequence $F = (F_n(T))_{n=1}^{\infty}$
- Irrationality of the dynamical zeta function suggests more complex and irregular behaviour of F
- A natural boundary implies even stronger results concerning the irregularity and complex behaviour of F compared to a dynamical zeta function which is irrational

The Dynamical Zeta Function

The dynamical zeta function has a natural boundary if it does not have an analytic continuation beyond its radius of convergence, where we define the radius of convergence, denoted by R, by

$$R = \frac{1}{\limsup_{n\to\infty} (F_n(T))^{1/n}}.$$

Then at $|z| = R$, we have a dense set of singularities or zeros.

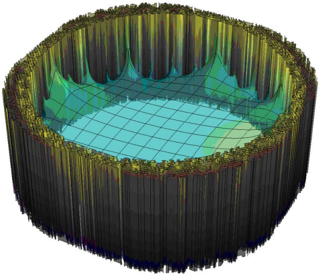


Figure 3: (Approximate) plot of the modulus of $\sum_{n=0}^{\infty} z^{2^n}$ showing a natural boundary at the unit circumference

The Dynamical Zeta Function– Irrationality as a Factor of Rationality

Let (Y, S) be a topological dynamical system, and let $G = \mathbb{Z}/2\mathbb{Z}$. Define $X = Y \times \{0, 1\}$ and $T : X \to X$ by

$$T(x, m) = (S(x), m + 1 \pmod 2)$$

Further, define an action of G by

$$g(x, m) = (x, m + 1 \pmod 2).$$

Then $(X', T') = (Y, S)$.

Now, take

$$F_n(T') = \begin{cases} 0, & \text{if } n = 1, \\ 2^n, & \text{if } n \text{ is even}, \\ \sum\limits_{\substack{d|n \\ d \neq 1}} d\, 2^{(d-1)/2}, & \text{if } n > 1 \text{ is odd}. \end{cases}$$

Then $\zeta_T(z)$ is given by

$$\zeta_T(z) = \frac{1}{1 - 4z^2},$$

which is rational.

The logarithmic derivative of $\zeta_{T'}(z)$ is given by

$$z\left[\frac{\zeta'_{T'}(z)}{\zeta_{T'}(z)}\right] = \sum_{n=1}^{\infty} z^n F_n(T') = \frac{4z^2}{1-4z^2} + \frac{6z^3-4z^5}{(1-2z^2)^2} + \alpha(z),$$

where

$$\alpha(z) = \sum_{k=1}^{\infty} a_k z^k$$

for

$$\begin{aligned} a_{2k} &= 0 \\ a_{2k+1} &= \sum_{\substack{d|2k+1 \\ d\neq 1 \\ d\neq 2k+1}} d\,2^{(d-1)/2}. \end{aligned}$$

Then $\alpha(z)$ (and therefore $\zeta_{T'}(z)$) is irrational.

We let $G = C_2$ and

$$F_n = 2^n \quad \text{and} \quad F'_n = 2^n + a_n 2^n,$$

where we define

- $a_n = 0$ if $n = 1$ or $n = p$, for any prime p
- $a_n \equiv a_{n/p} \pmod{p^{\text{order}_p(n)}}$ for any prime p such that $p \mid n$, where we choose a_n such that $n \leq a_n < 2n$.

Further, we set

$$b_n^{[H]_\sigma} = \begin{cases} \sum_{k=0}^{\text{order}_2(n)} \left(O'_{n/2^k} - O_{n/2^k} \right), & \text{if } [H]_\sigma = [I]_{(2,1)}, \\ O_n - b_n^{[I]_{(2,1)}}, & \text{if } [H]_\sigma = [G]_{(1,1)} \text{ and } n \text{ is even} \\ O_n, & \text{if } [H]_\sigma = [G]_{(1,1)} \text{ and } n \text{ is odd} \\ 0, & \text{otherwise} \end{cases}$$

and

$$\begin{aligned} a_n &= a_n^{[G]_{(1,1)}} + a_n^{[I]_{(2,1)}} \\ &= O_n, \end{aligned} \qquad \begin{aligned} b_n &= b_n^{[G]_{(1,1)}} + b_n^{[I]_{(2,1)}} \\ &= O'_n. \end{aligned}$$

The Dynamical Zeta Function– A Natural Boundary as a Factor of Rationality

By the Super Basic Lemma, there exists a topological dynamical system (X, T) and an action of C_2 on X which commutes with T such that

$$a_n = O_n(T) \qquad \text{and} \qquad b_n = O_n(T').$$

Then $\zeta_T(z)$ is given by

$$\frac{1}{1-2z},$$

which is rational, while $\zeta_{T'}(z)$ is given by

$$\left(\frac{1}{1-2z}\right) \exp\left(\sum_{n=1}^{\infty} \frac{a_n (2z)^n}{n}\right).$$

Now, if we set $f(z) = a_n z^n$, then

$$f(2z) = \sum_{n=1}^{\infty} a_n (2z)^n$$

and we find that $f(z)$ has a natural boundary at $z = 1$, so that $f(2z)$ (and therefore $\zeta_{T'}(z)$) has a natural boundary at $z = \frac{1}{2}$.

Open Questions

1. Windsor showed that there exists a smooth model (X, T) such that $O_n(T) = a_n$, for all $n \geq 1$. Then given sequences (a_n) and (b_n) as in the Super Basic Lemma, is there a smooth model (X, T) and an action of a finite group G on X which commutes with T such that $O_n(T) = a_n$ and $O_n(T') = b_n$, for all $n \geq 1$?
2. In the examples studied, both (X, T) and (X', T') exhibit the same asymptotic growth rates for orbits, and therefore, they are not representative for the general case of the Super Basic Lemma. We observe that the proof for the Basic Lemma uses a more abstract combinatorial construction of (X, T) very unlike the natural constructions of the examples studied. They occur in a natural setting: We have easy to define spaces and maps and the groups chosen are natural (non-trivial) choices for the given space and map. Do examples occurring in a natural setting, for example, classes of systems like group automorphisms, subshifts of finite type, expanding maps on an interval, always exhibit the same growth rates for orbits in the quotient system?

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