Periodic orbits in differential equations with state-dependent delay

Reminder: consider $\dot{x}(t) = \varphi(x(t), x(t-\tau))$, $\tau$ fixed $> 0$

Initial condition: $x(t)$ is a continuous function on $t \in [-\tau, 0] \rightarrow \mathbb{R}^n$

Example: $\dot{x}(t) = -x(t-x(t))$ is scalar, try & solve DE for $0 \leq t < \infty$

$x(0) = 1$, $x(-1) = -1$

$x(t) = 1 + t$ is solution: $\dot{x}(t) = 1$

$x(t-x(t)) = x(t-1-t) = x(-1) = -1$

When is $x(t) = 1 + t - t^2$ solution?

$\dot{x}(t) = 1 - 2t$

$x(t-x(t)) = x(t-1-t+t^2) = x(t-1) \rightarrow x(t-1) = 2t-1$

$s = t^2 - 1 \Rightarrow t = \sqrt{s+1} \Rightarrow x(s) = 2\sqrt{s+1} - 1$ for $-1 \leq s < 0$

Solution to IVP starting from continuous $x$ is not unique!

This is not a problem of causality ($x(t) < 0$, e.g.)

$\dot{x} = -x(t - \varphi(x(t)))$

Problem is initial slope at $t = -1$

Formulation of IVP: Let $x : [-\tau, t] \rightarrow \mathbb{R}^n$ be trajectory until now

$f : C([-\tau, 0]; \mathbb{R}^n) \rightarrow \mathbb{R}^n$ is right-hand-side

cont. func on $[-\tau, 0]$
The time shift: \( t \in \mathbb{R}, [\Delta t x](s) := x(t+s) \)

\( \Delta_t \) maps \( x: \mathbb{R} \to \mathbb{R}^n \) to \( \Delta_t x: \mathbb{R} \to \mathbb{R}^n \)

Then we can write general DDE as

\[ \dot{x}(t) = \phi(\Delta_t x) \]

in example \( \dot{x}(t) = \mu - x(t-x(\tau)) \)

\( \phi(x) = \mu - x(-x(\tau)) \)

\( \phi(x) = \phi(x(0), x(-\tau)) \)

\( \phi: U \subset C^0([\tau, 0]; \mathbb{R}) \to \mathbb{R} \) continuous, \( U = \{x : 0 < x(\tau) < 2\} \)

but not locally Lipschitz continuous

\( (\text{in } u \in N \text{ s.t. } x(t) - y(t) |\leq k \max_{t \in [\tau, 0]} |x(t) - y(t)| \text{ for } x, y \in V) \)

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\[ \phi(x) = \mu - x(x(t)) \]

\( \phi(x) = \phi(x(0), x(-\tau)) \)

\( \phi: C^k([\tau, 0]; \mathbb{R}) \to \mathbb{R} \) is \( k \) times cont. diff.

also \( \phi: C^{0,1}(\tau, 0], \mathbb{R}) \to \mathbb{R} \) is locally Lipschitz cont.

Possible phase space: solve \( \dot{x} = \phi(\Delta_t x), \quad x(s) = x(0) \text{ for } s \in [\tau, 0] \)

\( C^{0,1}([\tau, 0]; \mathbb{R}) \to \Delta_t x \text{ for } t > \tau \text{ depend } L\text{-cont. on } x_0 \)

\( U := \{x \in C^1([\tau, 0]; \mathbb{R}) : x(0) = \phi(x)\} \quad (x' = \mu - x(-x(t))) \)

nonlinear submanifold of \( C^1 \), codimension = 1 (generally 2)

condition prevents corner at \( t = 0 \)

\( \phi(\Delta t x) \)

semiflow \( U \to U \), \( \Delta t x \) depends cont. diff. on \( x_0 \)

in metric of \( U \subset C^1 \)

(H. O. Walther)

\( \to \) principle of linearized stability of bifurcations

general: \( U := \{x \in C^1([\tau, 0]; \mathbb{R}) : x'(0) = \phi(x)\} \)

upper bound for state-dependent delay
What about bifurcation analysis?

Find equilibria: easy! Example: \( 0 = \mu - x(t - x(t)) \), but \( x(t) \) is constant: \( x(t) = x_0 \) for all \( t \rightarrow 0 = \mu - x_0 \).

In general, equilibria defined as roots of system of \( n \) alg. eqs:

- constant extension: \( E_0 : \mathbb{R}^n \rightarrow C([-\tau,0];\mathbb{R}^n) \) \( (E_0)(t) := x_0 \)
- alg. system is \( 0 = \mathcal{F}(E_0 x_0) \) in eqs., \( n \) vars \( x_0 \)
- all static equilibrium bifurcation ok (fold, pitchforks, transcritical...)

Local stability of equilibria: lin. stability is ok
Linearization in \( E_0 x_0 \): linear DDE with constant delay
- example: \( x_0 = \mu \) \( x' = -x(t-\mu) \)
  - Eigenvalues \( \lambda = e^{-\mu}\lambda \) stable if \( \mu < \frac{\pi}{2} \)
  - \( i\omega = e^{-i\mu} \) loses stability at \( \mu = \frac{\pi}{2}, \omega = 1 \)

Hopf bifurcation?
- Similar result for periodic orbits: can be found as roots of system of algebraic eqs.

More precisely: consider periodic BVPs

\[ C^k(S;\mathbb{R}^n) := \left\{ x : [-\pi, \pi) \rightarrow \mathbb{R}^n, \text{k times cont. diff, } x^{(k)}(0) = x^{(k)}(\pi) \right\} \]

- \( S \) "unit circle"
  - \( \|x\|_S := \max \{ |x|, |x|', ... |x^{(k)}|' \} \)
- \( x \in C^k(S;\mathbb{R}^n) \) can be extended to \( \mathbb{R} : x(t) := x(t + \text{mod } [-\pi, \pi]) \)

General periodic BVP:
- nonlinear functional \( f : C(S;\mathbb{R}^n) \rightarrow \mathbb{R}^4 \) \( f(x) = -x(-x(0)) \)
- time shift \( \Delta_t : C(S;\mathbb{R}^n) \rightarrow C(S;\mathbb{R}^n) \) \( (\Delta_t x)(s) = x(s + t) \)
  - \( \dot{x}(t) = f(\Delta_t x) := F(x)(t) \) \( x \in C'(S;\mathbb{R}^n) \)
  - \( F : C(S;\mathbb{R}^n) \rightarrow C(S;\mathbb{R}^n) \)
For our example:
\[
\begin{align*}
\dot{x}(t) &= \frac{1}{\omega} \left[ p - x(t - \omega x(t)) \right] \\
\dot{\omega}(t) &= 0 \\
\dot{\rho}(t) &= 0
\end{align*}
\]

\[\text{Ext} = \left\{ \left( \begin{array}{c} X \\ \omega \\ \rho \end{array} \right) \in C^1(S; \mathbb{R}^3) \mid \frac{2\pi}{\omega} - \text{Period of orbit} \right\}
\]

\[\text{Ext} \left( \begin{array}{c} X \\ \omega \\ \rho \end{array} \right) = \left[ \begin{array}{c} \frac{1}{\omega} [\rho(t) - x(t - \omega(t)x(t))] \\ 0 \\ 0 \end{array} \right]
\]

Main result:

\[\dot{x}(t) = F(x(t)) = \frac{d}{dt} f(\Delta x(t)) \iff \text{if and only if } \quad g(p) = 0 \quad \text{and } x = \lambda(p)
\]

where \( g: \mathbb{R}^m \to \mathbb{R}^n \) is \( C^k \), \( p \in \mathbb{R}^m \) and \( x: \mathbb{R}^m \to C^0(S; \mathbb{R}^n) \)

\( m \) depends on "local Lipschitz constant" of \( F \)

\( k \) depends on "smoothness" of \( F \) (in example \( k = \infty \))

equivalence is valid locally (in nbh of \( x_0 \in C^\infty(S; \mathbb{R}^n) \))

\( \Rightarrow \) Hopf bif!

(lack of) regularity of \( f \) and \( F(x) = f(\Delta x, x) \)

Example:
\[ f(x) = -x(x(0)) \quad \text{(ignore } \rho, \omega) \quad F(x)(t) = -x(t + x(t)) \]
\[ \partial x f(x) y = x'(x(t)) y(t) - y'(x(t)) \]
\[ \partial F f(x) y = x'(t + x(t)) y(t) - y'(t + x(t)) \]

\( f \) is \( C^k \) only as a map \( f: C^k(S; \mathbb{R}^n) \to \mathbb{R}^n \)
\( F \) is \( C^k \) only as \( F: C^{2+k}(S; \mathbb{R}^n) \to C^0(S; \mathbb{R}^n) \)

\( F: C^k \to C^k \) is only continuous, not locally Lipschitz cont.
Basic idea for construction of algebraic system

Poincaré map not useful :: still an infinite-dimensional map  
  at most C¹

Alternative:

\[
\dot{x} = F(x)(t) = f(x,t) \quad \Rightarrow \quad x(t) = x(0) + \int_0^t F(x)(s) \, ds
\]

does not map \( C(S) \rightarrow C(S) \)

Look at this:

\[
x(t) = p + \int_0^t F(x)(s) - P_0 F(x)(s) \, ds \quad \text{ (FP)}
\]

\[P_0 y = \text{avg of } y\]

\(p \in \mathbb{R}^5\) parameter

If \( x \mapsto E_0 p + \int_0^t F(x)(s) - P_0 F(x)(s) \, ds \) is contraction  \(\Rightarrow\) (FP) has unique soln:

\[X(t) \quad p \in \mathbb{R}^5 \quad X(t) \in C(S; \mathbb{R})\]

alg. system:

\[P_0 F(X(t)) = 0\]

But: \( F(x)(t) = -x(t - \frac{\pi}{2}) + \int_0^t x(s - \frac{\pi}{2}) \, ds \) has EV 1, Evec sin t

Generalize:

\[P_N y = \text{proj of } y \in C(S; \mathbb{R}^n) \text{ onto first } N \text{ Fourier modes}\]

\[(P_N y)_i(t) = \sum_{-N}^N \langle b_k, y \rangle b_k(t) \quad \langle b_k, y \rangle = \frac{1}{\pi} \int_{-\pi}^\pi b_k y \, ds \quad b_k = \frac{1}{\pi} \cos(k t), b_{-k} = \sin(k t)\]

\[O_N y = y - P_N y, \quad P_N y = \text{average of } y\]

\[R_N : C^k(S, \mathbb{R}^n) \rightarrow \mathbb{R}^{(2^N n)} \quad R_N g_i = \langle b_k, y_i \rangle \quad k = -N, \ldots, N\]

\[P_i (s + t) = \sin t, \quad R_i (s + t) = [0]\]

\[E_N : \mathbb{R}^{(2^N n)} \rightarrow C^k(S; \mathbb{R}^n) \quad (E_N p)(t) = \sum_{i=0}^N P_i p_i b_k(t) \quad i.e. \{\cdot\}\]

\[P_N = E_N R_N\]
FP problem: \[ x(t) \in \arg\min_{x \in \mathbb{R}} \left\{ \frac{1}{2} x_0^T F(x)(t) x_0 \right\} \] (FPP)

is parametric fixed point problem (par p)

if F is loc. Lipschitz continuous: \[ \|F(x) - F(y)\| \leq K \|x - y\| \]
for all \( x, y \in \mathbb{R} \) and \( p \in \text{int}B \) \rightarrow \( (FPP) \) has unique solution \( x = X(p) \)

algebraic system for \((p,c)\): \[ 0 = \sum_p F(X(p)) \] & any of time derivative \[ c = E_N p - P_N \int_0^T F(X(p))(s) ds \] & \( p \in \text{int}B \)

is constant in \( \mathbb{R}^4 \), \( u(\text{2ms}) \) vars, \( u(\text{2ms}) \) eqs

restricted loc. Lipschitz continuity & differentiability

\[ f(x) = -x(x(t)) \]
\[ \partial f(x) = x'(x(t)) y(t) - y'(x(t)) \] does not depend on \( y' \)

\[ \partial f : C \rightarrow \mathbb{R}^4 \]
\[ \partial f : C \times C \rightarrow \mathbb{R}^4 \] continuous
\[ \lim_{y \rightarrow 0} \frac{|f(x+y) - f(x) - \partial f(x)y|}{|y|} = 0 \] for all \( x, y \in C' \)

implies: \[ |f(x) - f(y)| \leq K \|x - y\| \] for all \( x, y \) in sufficiently small \( N_b \) \( N_1(0) \) \( K \in C^1 \)

also: for all \( x, y \) in sufficiently small \( N_b \) \( N_1(0) \) \( K \in C^1 \)

closed balls in \( C^0 \) are complete wrt. \( \|\cdot\|_0 \) norm.

so we can apply BCMP to \( B \in C^{0,1} \) with norm \( \|x\|_0 = \max |x(t)| \)