Numerical problems with delay-differential equations (DDEs)

AUTO, MatCont, COCO, XPPAUT ⇝ bifurcation diagrams for ODEs

$\dot{x}(t) = f(x(t), p)$, $x(t) \in \mathbb{R}^n$

DDE-Biftool ⇝ for DDEs (KU Leuven: Engelborghs, Roose, Luzyanina)

Simple case $M \dot{x}(t) = f(x(t), x(t-\tau), p)$, $\tau > 0$, M $n \times n$ possibly singular

track equilibria + linear stability & bifurcations

periodic orbits + linear stability & bifurcations

Problems occur for even simple-looking problems

exponential stability of DDE w. constant coefficients:

$(M = I)$ $\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau)$ | ODE: $\dot{x}(t) = A_0 x(t)$

↳ compute eigenvalues λ of A_0 $(n \times n)$

DDE ≡ PDE (transport equation)

ODE: $\dot{x}(t) = A_0 \tilde{x}(t, 0) + A_1 \tilde{x}(t, -\tau)$

PDE: $\partial_t \tilde{x}(t,s) = \partial_s \tilde{x}(t,s)$ for $s \in [-\tau, 0]$

BC: $\tilde{x}(t, 0) = x(t)$

Solution $x(t), \tilde{x}(t, \cdot)$ ⇝ Phase space $\mathbb{R}^n$, $C([-\tau, 0]; \mathbb{R}^n)$ ⇐ continuous functions on $[-\tau, 0]$

Stability: solve eigenvalue problem

E_0 $\lambda x = A_0 \tilde{x}(0) + A_1 \tilde{x}(-\tau)$

E_s $\lambda \tilde{x}(s) = \partial_s \tilde{x}(s)$ ⇒ $\tilde{x}(s) = e^{\lambda s} x$ ⇒ $\lambda x = A_0 x + A_1 e^{-\lambda \tau} x$

BC: $\tilde{x}(0) = x$

↳ λ Eigenvalue ⟺ $\boxed{\underbrace{\det[\lambda I - A_0 - A_1 e^{-\lambda \tau}]}_{d_\infty(\lambda)} = 0}$ ⇐ not useful for numerical computations (only for checking)

Approximation: use polynomial x_N of degree N, impose ⇒ $n \times (N+1)$ variables
(Breda)

$\lambda x_N(0) = A_0 x_N(0) + A_1 x_N(-\tau)$ $x_N(0), x_N(s_1), \dots, x_N(s_N)$

$\lambda x_N(s) = x_N'(s)$ at $s_1, \dots, s_N \in [-\tau, 0]$ (Chebyshev nodes)

some interpolation necessary if multiple delays

↳ $\lambda I x_N = \underbrace{\begin{bmatrix} A_0 & \cdots & A_1 \\ - & D & - \end{bmatrix}}_{\mathcal{A}_N} x_N$ $\begin{matrix} \}n \\ \}Nn \end{matrix}$ (E_N)

↳ eigenvalue problem for big matrix $\mathcal{A}_N$

Basic demo: $\dot{x}(t) = -x(t-\tau)$, $\rightsquigarrow A_0 = 0,\ A_1 = -1$,

if τ larger $\rightsquigarrow$ problem more difficult

$\boxed{\text{Theoretical statement (Breda, Vermiglio 2005)}}$ Construction of discrete characteristic function

For given $\lambda \in \mathbb{C}$, $x_0 \in \mathbb{C}^s$, define the unique polynomial of degree N $p_N(s;\lambda,x_0)$ by

$p_N(0) = x_0$

$p_N'(s_j) = \lambda p_N(s_j)$ at N nodes s_j in $[-\tau, 0)$, $j = 1..N$

Then $(x_0,\dots,x_N) = (p_N(0), \dots p_N(s_N))$ satisfies (F_N) $\Leftrightarrow$

$$\lambda x_0 = A_0 x_0 + A_1 p_N(-\tau;\lambda,x_0)$$

$\hookrightarrow$ Define $\boxed{d_N(\lambda) = \det\left[\lambda I_n - A_0 - A_1 p_N(-\tau;\lambda,I_n)\right]}$

Theorem (Breda & Vermiglio '05): $|d_\infty(\lambda) - d_N(\lambda)| \leq C_2 \frac{1}{\sqrt{N}} \left(\frac{C_1}{N}\right)^N$

Corollary:

Let λ_∞ be a root of $d_\infty(\lambda) = \det[\lambda I - A_0 - A_1 e^{-\lambda\tau}]$ of multiplicity k.

Then $d_N(\lambda) = \det[\lambda I - A_0 - A_1 p_N(-\tau;\lambda,I)]$ has k roots $\lambda_{1,N},\dots,\lambda_{k,N}$

and $|\lambda_{i,N} - \lambda_\infty| \leq C_3 \left[\frac{1}{\sqrt{N}} \left(\frac{C_1}{N}\right)^N\right]^{1/k}$

Example: $\dot{x}_1(t) = \lambda_1 x_1(t) + \varepsilon x_2(t-\tau)$, $\lambda_1 \neq \lambda_2 < 0$

$\dot{x}_2(t) = \lambda_2 x_2(t)$ $\tau = 20$, $\varepsilon \sim \frac{1}{\tau}$

$$d_\infty(\lambda) = (\lambda-\lambda_1)(\lambda-\lambda_2)$$
$$= \det\begin{bmatrix} \lambda-\lambda_1 & -\varepsilon e^{-\lambda\tau} \\ 0 & \lambda-\lambda_2 \end{bmatrix}$$

$\approx 10^5$

Suggestion for source of problems

Introduce small perturbation $\dot{x}_2 = \lambda_2 x_2(t) + \delta x_1(t-\tau)$

many new eigenvalues, very different (and dominant ones are correct)

Eigenvalue of $\mathcal{A}_\infty$ at $-\infty$ has multiplicity ∞.

($\exp(\mathcal{A}_\infty)$ has EV 0 of multiplicity ∞.)

$\Rightarrow$ $\mathcal{A}_N$ has $\sim N$ eigenvalues that are in uncontrollable places.

$\Rightarrow$ these uncontrolled eigenvalues of $\mathcal{A}_N$ interfere with λ_∞.