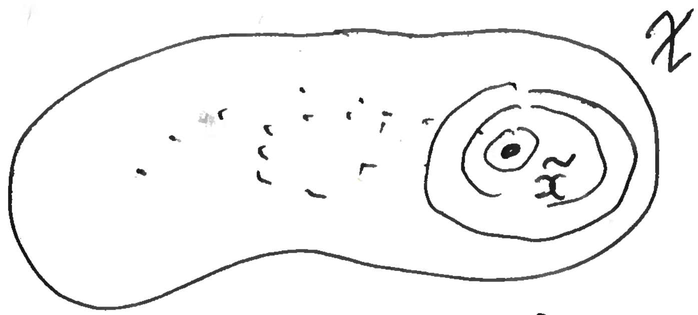
1) Almost sure growth of maxima

m.p.t $f : \mathcal{X} \to \mathcal{X}$, prob space $(\mathcal{X}, \mu, \mathcal{B})$.

Process $X_n(x) = \phi(f^{n-1}x)$, $x \in \mathcal{X}$,

observable $\phi : \mathcal{X} \to \mathbb{R}$.

Maxima process $M_n(x) = \max_{k \le n} X_n(x)$.



Take $\psi : [0, \infty) \to \mathbb{R}$.

$\phi(x) = \psi(\mathrm{dist}(x, \tilde{x}))$

e.g. $\psi(z) = -\log z$, $z^{-\alpha}$, $\sup \phi = \infty$.

$\partial B(\tilde{x}, r) = \{x : \phi(x) = \psi(r)\}$.

- If f ergodic, $M_n \to \infty$, μ-a.e $x \in \mathcal{X}$.
- Determine u_n, v_n such that for μ-a.e $x \in \mathcal{X}$, $v_n \le M_n(x) \le u_n$, $\forall n \ge N_x$.
- Recall the strong-BC property.

Let $S_n(x) = \sum_{k=1}^{n} 1_{B_k}(f^{k-1}x)$, targets $(B_k)_{k\geq 1}$. Let $E_n = \sum_{k=1}^{n} \mu(B_k) \longrightarrow \infty$.

Then for μ-a.e $x \in \mathcal{X}$, $\dfrac{S_n(x)}{E_n} \longrightarrow 1 \quad (n \to \infty)$.

<u>Easy case</u>: Suppose $u_n \nearrow \infty$ monotone.

Then $\mu(M_n \geq u_n, \text{ i.o.}) = \mu(X_n \geq u_n, \text{i.o})$.

"i.o" = infinitely often.

If $u_n = \psi(r_n)$, $\sum_{n=1}^{\infty} \mu(B(\tilde{x}, r_n)) < \infty$, then

$$\text{BCI} \Rightarrow \mu(X_n \geq u_n, \text{i.o}) = 0.$$

$$\Rightarrow \mu(M_n \geq u_n, \text{i.o}) = 0$$

$$\Rightarrow \mu(M_n < u_n, \text{ eventually}) = 1.$$

e.g. r_n : $\mu(B(\tilde{x}, r_n)) \leq \dfrac{1}{n(\log n)^{1+\varepsilon}}$

Lower bound.

let $L = \sup\{k \leq n : f^k(x) \in B(\tilde{x}, r_k)\}$.

$$\sum_{k=1}^{\infty} \mu(B(\tilde{x}, r_k)) = \infty.$$

$L \equiv L(n) \to \infty, \quad n \to \infty.$

Using SBC: $S_n(x) = S_L(x)$.

$$|S_n(x) - E_n| = o(E_n).$$

$$|S_L(x) - E_L| = o(E_L) \quad , E_L \leq E_n.$$

$$E_n = \sum_{k=1}^{n} \mu(B(\tilde{x}, r_k)).$$

$$\Rightarrow \quad E_L = (1+o(1))E_n.$$

We get $M_n(x) \geq M_L(x) = \Psi(r_{L(n)})$.

E.g. $\mu(B(\tilde{x}, r_n)) = \frac{1}{n}$.

Then $E_n \sim \log n \quad , E_L \sim \log L$.

$$\Rightarrow \log L = (1+o(1)) \log n.$$

$$\Rightarrow \forall \varepsilon > 0, \quad L \geq n^{1-\varepsilon}.$$

$$\Rightarrow M_n(x) \geq \Psi(r_{n^{1-\varepsilon}}),$$

$$\mu(B(\tilde{x}, r_{n^{1-\varepsilon}})) = \frac{1}{n^{1-\varepsilon}}.$$

- Balls of measure $\frac{1}{n(\log n)}$.

However $\log\log L = (1+o(1)) \log\log n$.

Using this SBC approach:

$M_n(x) \geq V_n$ eventually, μ-a.e $x \in \mathcal{X}$,

where $V_n = \psi(r_n)$, $\mu(B(\tilde{x}, r_n)) \geq \frac{(\log n)^\beta}{n}$.

$\beta > 2$.

- Kim 2007
- Gupta-Nicol-Ott 2010.
- H, Nicol, Török 2016.
- Kirsebom, Kunde, Persson 2020.

I.I.D. Klass 1984, 1985, Galambos.

$(X_n)_{n \geq 1}$, distribution function

$$F_{X_1}(x) = \mu(X_1 \leq x), \quad \bar{F}_{X_1}(x) = \mu(X_1 \geq x).$$

- If $\bar{F}_X(u_n) \geq \frac{c \log\log n}{n}$ for $c > 1$,

  then $\mu(M_n \geq u_n, \text{ eventually}) = 1$.

- If $\bar{F}_X(u_n) \leq \frac{c \log\log n}{n}$ for $c < 1$

  $\mu(M_n < u_n, \text{i.o}) = \mu(M_n > u_n, \text{i.o}) = 1$.

  $$\sum_{n=1}^{\infty} \bar{F}_X(u_n) = \infty$$

- Continued fractions: Philipp 1976.

$$x = [a_0(x) \cdots a_n(x) \cdots].$$

Then $\liminf_{n\to\infty} \frac{\log\log n}{n} \operatorname{Max}_{k \le n} a_k(x) = \frac{1}{\log 2}$.

μ-a.e $x \in [0,1]$.

- <u>Caution</u> : $(X_n)_{n\geq 1}$ i.i.d.

$$Y_n = \operatorname{Max}(X_n, X_{n+1}).$$

Corresponding c as stated in the klass conditions for $F_Y(u_n)$ is 2.

$$\operatorname{Max}(Y_1 \cdots Y_n) = \operatorname{Max}(X_1 \cdots X_{n+1}).$$

$$F_Y(u) = F_X(u)^2$$

$$\bar{F}_Y(u) = 2\bar{F}_X(u) + \text{Error}.$$

Distributional convergence of maxima.

Set up $(\mathcal{X}, \mu, \mathcal{B})$, $f: \mathcal{X} \to \mathcal{X}$.

$$M_n(x) = \max_{k \leq n} \phi(f^{k-1} x), \quad x \in \mathcal{X}. \qquad \phi: \mathcal{X} \to \mathbb{R} \text{ observable.}$$

$\mu(M_n \leq u_n)$. Determine sequences a_n, b_n and a limit function G such that

$$\lim_{n \to \infty} \mu(a_n(M_n - b_n) \leq u) = G(u).$$

<u>Theorem</u>: Consider (X_n) i.i.d, Given $\tau > 0$ let u_n be such that $n\mu(X_1 \geq u_n) \to \tau$. $(n \to \infty)$. (i.e $u_n \equiv u_n(\tau)$). Then:

$$\mu(M_n \leq u_n) \to e^{-\tau} \quad (n \to \infty),$$

and conversely.

<u>proof</u>:
$$\begin{aligned}
\mu(M_n \leq u_n) &= \mu(X_1 \leq u_n, \cdots X_n \leq u_n) \\
&= \mu(X_1 \leq u_n)^n \\
&= (1 - \mu(X_1 \geq u_n))^n \\
&= \left(1 - \frac{\tau}{n}(1 + o(1))\right)^n \to e^{-\tau}.
\end{aligned}$$

E.g. $Y_n = \mathrm{Max}(X_n, X_{n+1})$, $(X_n)_{n\geq 1}$ i.i.d.

Choose u_n: $n\bar{F}_Y(u_n) \to \tau$.

Then $F_Y(u_n) = F_X(u_n)^2$

$$
\begin{aligned}
\bar{F}_Y(u_n) &= \mu(Y > u_n) \\
&= 1 - (1 - \bar{F}_X(u_n))^2 \\
&\sim 2\bar{F}_X(u_n)
\end{aligned}
$$

$$\Rightarrow \mu(M_n^Y \leq u_n) \xrightarrow[n\to\infty]{} e^{-\tau/2}. \quad \theta = \frac{1}{2}$$

__Def__: Suppose $n\mu(X_1 > u_n) \to \tau$. We say $(X_n)_{n\geq 1}$ has extremal index θ if

$$\mu(M_n \leq u_n) \to e^{-\theta\tau} \quad \forall \tau > 0.$$

__Remark__: $\theta \in [0,1]$.

- Dependent processes. (__Blocking method__).

  Leadbetter. et. al 1983.

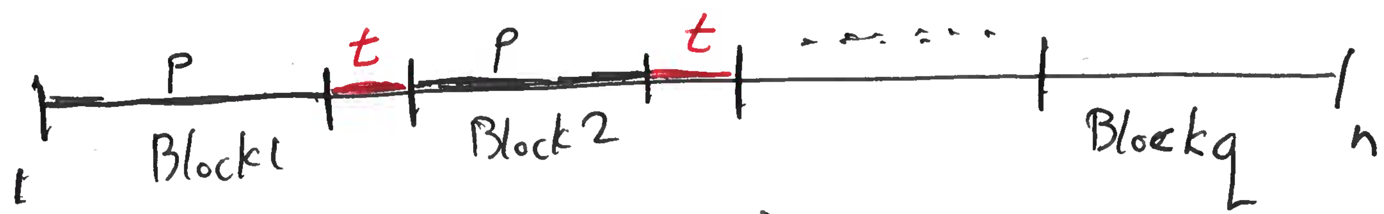
  Collet 2001.

  Lucarini. et. al 2016.

- $X_n = \phi \circ f^{n-1}$, $\quad f : \mathcal{X} \to \mathcal{X}$.

n, $(p, q, t \to \infty,\ n \to \infty)$. $\quad p, q \sim \sqrt{n}$

$n = q(p+t)$. $\quad t \sim (\log n)^2$.



In block 1: $\mu(M_p \leq u_n) = 1 - p\mu(X_1 \geq u_n)$ + Error.

$$\left| \mu(M_n \leq u_n) - \underline{\left(1 - p\mu(X_1 \geq u_n)\right)^q} \right| \leq \underline{\varepsilon(p,q,t)}$$

- $$(1 - p\mu(X_1 \geq u_n))^q = \exp\{q \log(1 - \mu(X_1 \geq u_n)p)\}$$
$$= \exp\{-pq\mu(X_1 \geq u_n)\}(1 + o(1))$$
$$\sim \exp\{-n\mu(X_1 \geq u_n)\} \to e^{-\tau}.$$

- Correlation decay. Error depends on $C(p,q)\alpha^t$, $\alpha \in (0,1)$.

- Error contribution:

$$n \sum_{j=2}^{p} \mu(X_1 \geq u_n, X_j \geq u_n). \quad (*)$$

Problem if: $\mu(X_1 \geq u_n, X_j \geq u_n) \approx \frac{1}{n}$.

want $(*) \to 0$ as $n \to \infty$. $D'(u_n)$.

What about $a_n, b_n, G(u)$?

$f(x) = 2x \bmod 1$, $\mu = \text{Leb}$.

$\phi(x) = -\log \text{dist}(x, \tilde{x})$. $\quad \left(u_n = \frac{u}{a_n} + b_n\right)$

$$n\mu\left(X_1 \geq \frac{u}{a_n} + b_n\right) = n \,\text{Leb}\left\{ d(x,\tilde{x}) \leq e^{-\frac{u}{a_n}} e^{-b_n} \right\}$$

Choose $a_n = 1$, $b_n = \log n + \log 2$

$$= e^{-u}$$

Then (Collet 2001), $\mu(M_n \leq u + \log 2 + \log n)$

For μ-a.e $\tilde{x} \in \mathcal{X}$ $\quad \to \exp\{-e^{-u}\}$.

Gumbel distribution. $u \in (-\infty, \infty)$.