Introduction to plasma physics: gyration, drifts, plasma oscillations, elements of plasma kinetics

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Part 1 – What is plasma?

In physics plasma is an ionized gas, and is usually considered to be a *distinct phase of matter*.

The free electric charges make the plasma *electrically conductive* so that it *couples strongly to electromagnetic fields*.

This fourth state of matter was first identified by Sir William Crookes in 1879 and dubbed “plasma” by Irving Langmuir in 1928, because it reminded him of a blood plasma.

The term plasma is generally reserved for a system of charged particles large enough to behave as one.

Even a partially ionized gas in which as little as 1% of the particles are ionized can have the characteristics of a plasma (i.e. <u>respond to EM fields</u> and be <u>highly electrically conductive</u>).

The two basic necessary (but not sufficient) properties of the plasma are:

a) Presence of freely moving charged particles, and
b) Large number of these particles (neutrals may be also present).

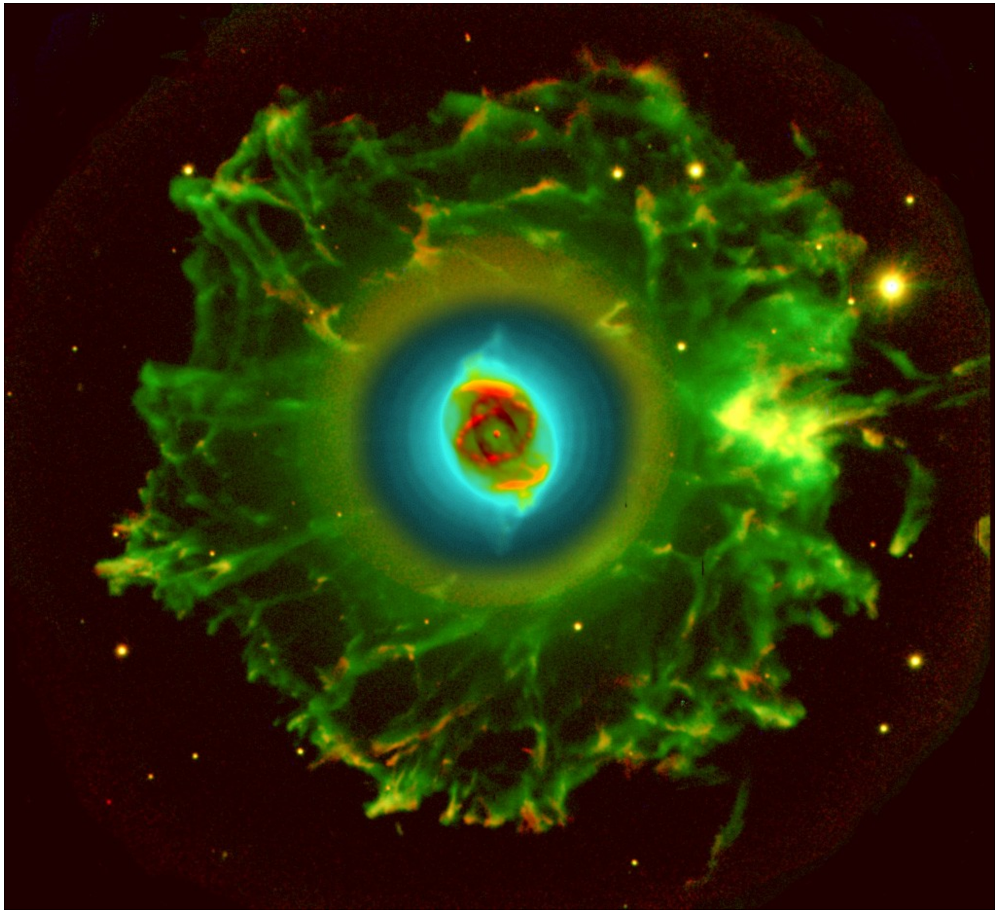
Property	Gas	Plasma
Electrical Conductivity	**Very low** (zero)	**Very high** 1) quasi-neutrality: to a good approx. $n_e = Zn_i$, 2) Possibility of currents couples the plasma strongly to EM fields.
No of species	**One**	**Two or three**
Velocity distribution	**Maxwellian**	**May be non-Maxwellian**
Interactions	**Binary**	**Collective** Each particle interactssimultaneously with many others. These collective interactions are much more important than binary collisions.

Major Plasma Sources

Plasmas are the most common phase of matter.

Over 99% of the visible universe is known to be plasma, such as that seen here in the Cat's Eye Nebula, and making up all

* the stars,
* much of the interstellar space between them, and
* the inter-galactic space between galaxies.



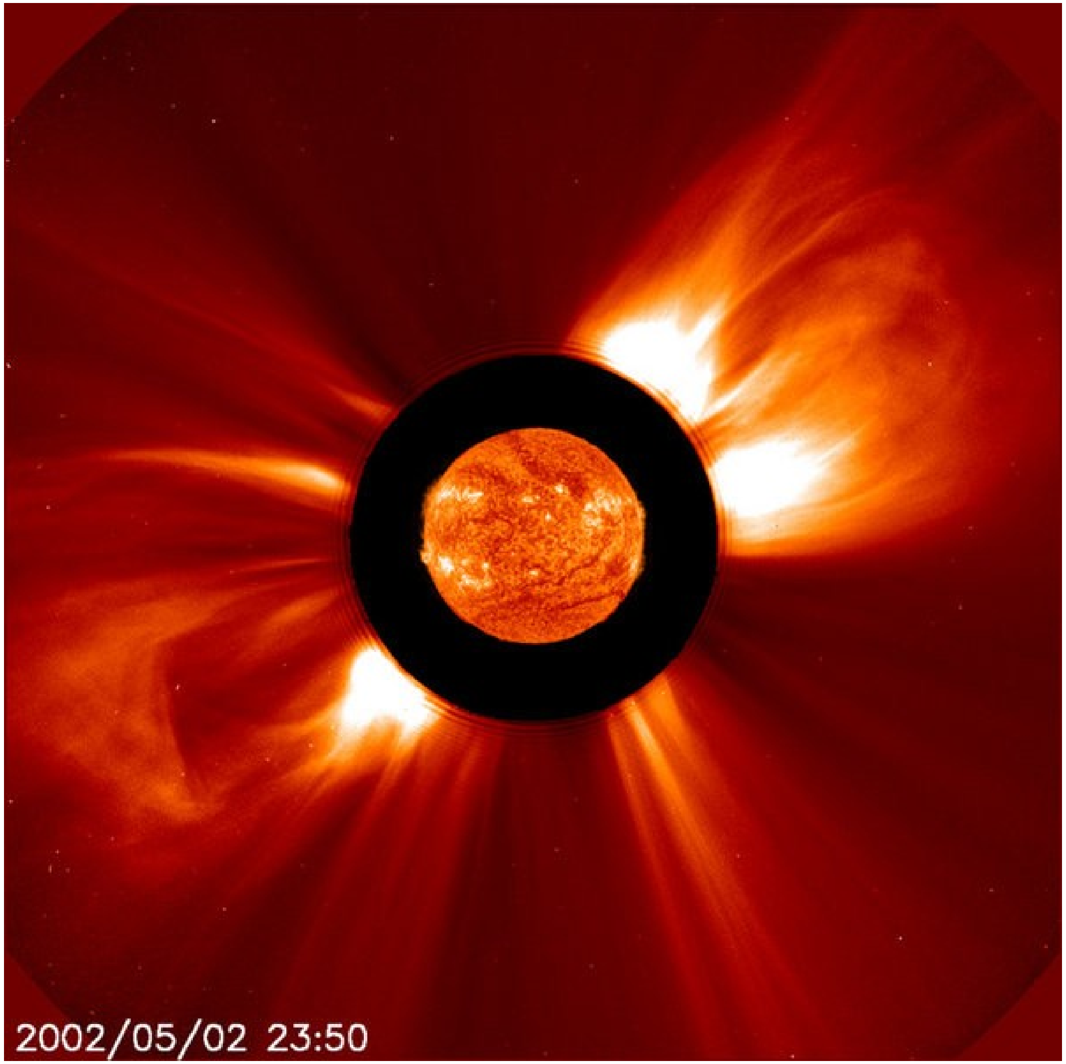
The sources of *space* plasmas are:

* The Sun (interior);

* The solar wind (continuously expanding atmosphere);

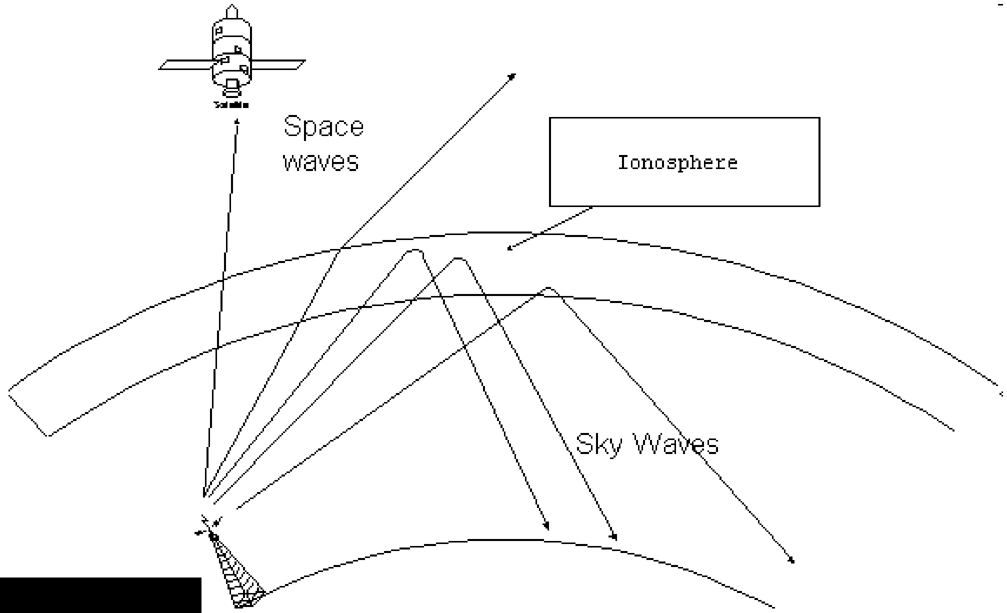
* The interplanetary medium;

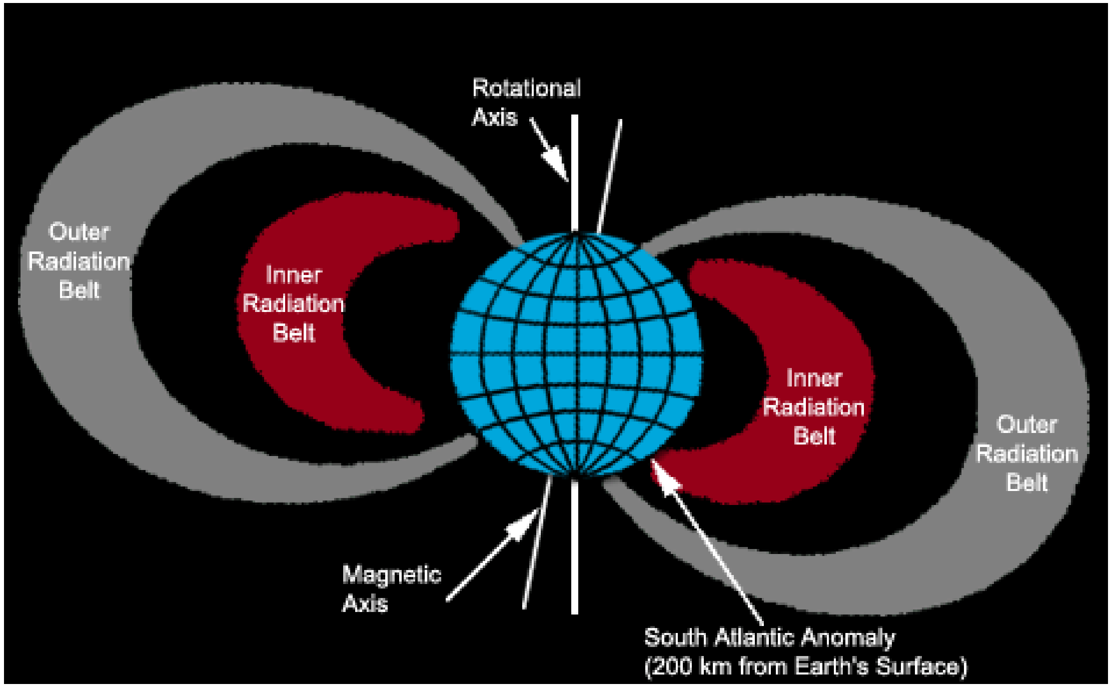
* Accretion disks; (during planetary formation).



Around *earth* plasmas:

* the ionosphere
(the part of the atmosphere that is ionized by solar radiation).



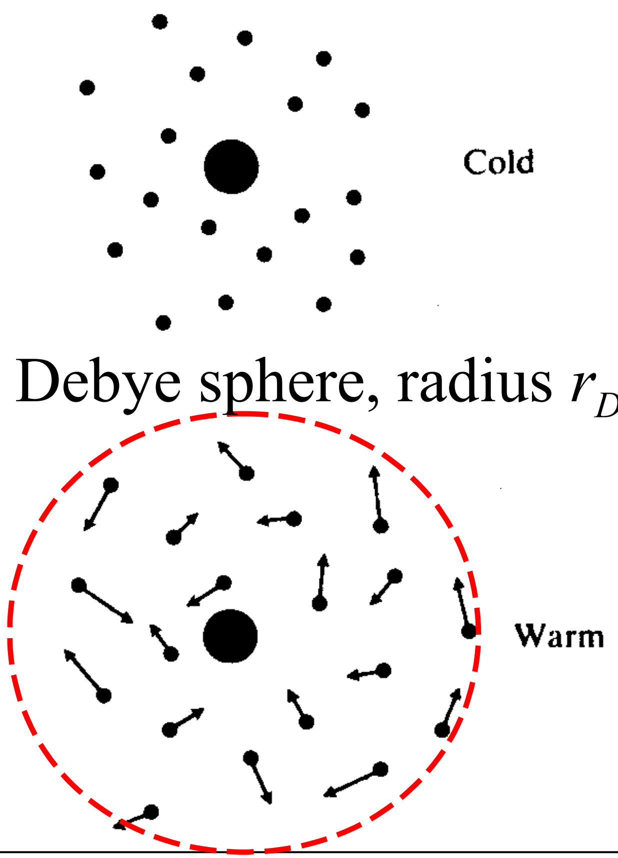


* Van Allen radiation belts

*magnetosphere

Debye screening (shielding) – **collective effects**, example 1

Consider one of the examples of collective effects in plasma – *Debye screening* which occurs when a test, +Q charge is introduced at $r = 0$ into a warm plasma:



Poisson equation for electric field for ($r > 0$):

$$\mathrm{div}E = \rho_e / \varepsilon_0 \qquad (1)$$

(This is one of the Maxwell's Equations)
Here, ρ_e is the charge density:

$$\rho_e = e(n - n_0)$$

Here n is the number density (number of particles per unit volume) of the freely moving charges in the presence of the test charge, while n_0 is their number density in the absence of this charge.

Next steps

1. Instead of electric field we introduce a electric potential $\vec{E} = -\nabla \phi$

2. Assume that number density obeys the Boltzmann law (simplified):

$$n = n_0 \exp\left(-U/kT\right) = n_0 \exp\left(-e\phi/kT\right) \approx n_0\left(1 - e\phi/kT\right)$$

because we can use Taylor expansion exp(ε) ≈ 1+ε when ε <<1.

Obtaining,

$$\rho_e = en - en_0 = en_0 - \frac{n_0 e^2}{kT}\phi - en_0 = -\frac{n_0 e^2}{kT}\phi$$

Substituting this back into Eq.(1) yields:

$$\nabla^2 \phi = \frac{n_0 e^2}{\varepsilon_0 kT}\phi = \frac{1}{r_D{}^2}\phi \qquad (2)$$

Here we use notation

$$\vec{\nabla} = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

Here we introduced a new physical quantity which has units of length:

$$r_D = \sqrt{\frac{\varepsilon_0 kT}{n_0 e^2}}$$

This known as the **Debye radius** (we will see why on the next slide).

It is easy to show (e.g. by substitution) that Eq.(2) has the following solution:

$$\boxed{\phi = \frac{Q}{4\pi\varepsilon_0 r}\exp\left(-r/r_D\right) \qquad (3)}$$

which is known as the *Debye screening (shielding) potential*.

Let us consider two limiting cases:

1. $r << r_D$ then $\qquad \phi \approx \frac{Q}{4\pi\varepsilon_0 r}$

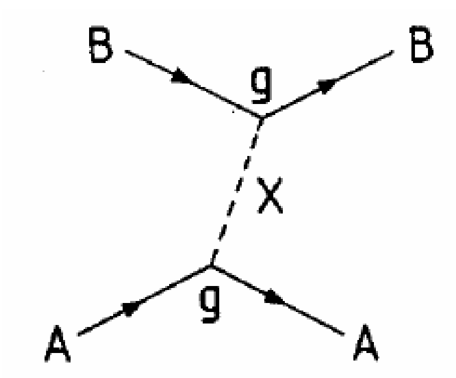
2. $r >> r_D$ then $\qquad \phi \propto \exp\left(-r/r_D\right)$

It is said that the plasma charges effectively *screen out* or *shield* the electric field of the test charge outside of the Debye sphere $r = r_D$. The phenomenon is called *Debye screening* or *shielding*.

Yukawa model of strong (nuclear) interaction

It turns out that Debye screening plasma potential has some interesting analogies in particle physics.

In particle physics, the interaction (strong, electromagnetic or weak) between particles is pictured by exchanging virtual particles which carry that interaction.



In order to create this virtual particle we need to borrow at least $\Delta E = M_X c^2$ amount of energy from the physical vacuum.

Heisenberg uncertainty principle sets limit on time interval Δt, during which this energy conservation violation is allowed: $\Delta E \Delta t \approx \hbar$. Thus, we have

$$\Delta t \approx \hbar / \Delta E \approx \hbar / (M_X c^2)$$

In order to estimate range of interaction, it is conservative to recall that no information can be communicated faster than the speed of light:

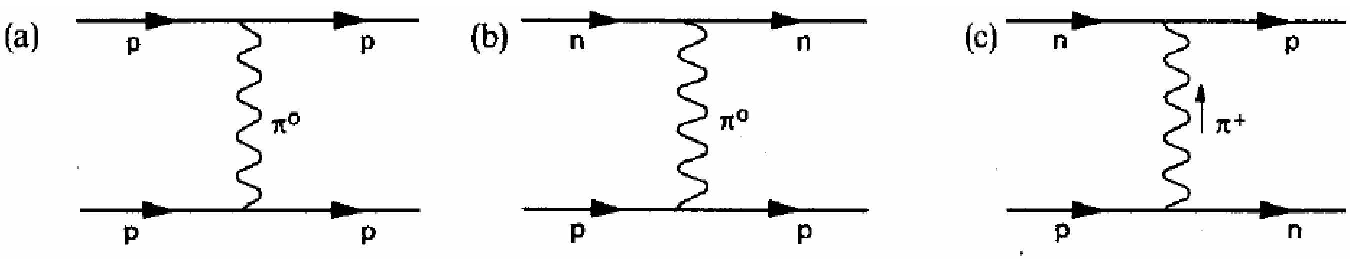
$$R = c\Delta t \approx \hbar c/(M_X c^2)$$

This formula asserts that the range of electromagnetic interaction is infinite because photons are massless: $R \to \infty$ for $M_X \to 0$.

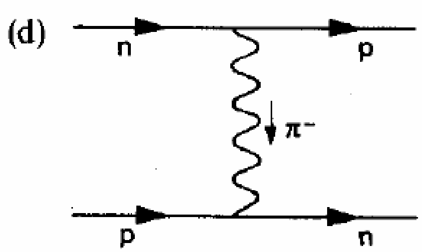
Also since empirically we know the range of strong (nuclear) interaction is about 1 Fermi (1 fm = 10^{-15}m), we can easily estimate what should be the mass of strong nuclear force career exchange particle:

$$R \approx \hbar c/(M_X c^2) \approx 200\text{MeV} \cdot fm/(M_X c^2) = 1\,fm \Rightarrow M_X \approx 200\text{MeV}/c^2$$

In fact, such particles are discovered and are called π-mesons. "meso" in Greek means middle referring to the mass that is in between an electron 0.511 MeV/c^2 and 940 MeV/c^2 for a nucleon (common name for a proton or a neutron).



Feynman diagrams of nucleon-nucleon interaction.



There are three types of π-mesons (also called pions): with electric charges -1, 0, +1 to account for mutual conversion of protons into neutrons. Their measured masses are $\pi^{\pm}$= 140 MeV/c^2 and π^0=135 MeV/c^2

Hideki Yukawa used simple relativistic relation between energy, *E*, momentum, *p*, and rest energy (which is true for any physical system):

$$E^2 = p^2c^2 + M_X^2c^4$$

In quantum mechanics each physical quantity has a corresponding quantum operator:

$$E \rightarrow i\hbar\frac{\partial}{\partial t} \text{ and } \vec{p} \rightarrow -i\hbar\vec{\nabla}$$

Making these substitutions yields

$$-\hbar^2\frac{\partial^2\phi}{\partial t^2}=-\hbar^2c^2\nabla^2\phi+M_X^2c^4$$

which is a relativistic, wave-like equation called Klein-Gordon equation for the interaction potential.

In the static case ($\partial/\partial t = 0$), the latter reduces to

$$\nabla^2\phi=\frac{M_X^2c^2}{\hbar^2}\phi=\frac{1}{R^2}\phi \qquad (4)$$

which is mathematically equivalent to Eq.(2) for the Debye screening plasma potential!

Note that for M_X =0 Eq.(4) gives simple electrostatic potential of a point charge:

$$\phi=\frac{Q}{4\pi\varepsilon_0 r}$$

while, full solution ($M_X \neq 0$) is

$$\boxed{\phi = \frac{const}{r} \exp\left(-r/R\right) \qquad (5)}$$

which is similar to Debye screening plasma potential. Thus, we conclude that plasma collective, electromagnetic interactions bear strong similarities with the strong nuclear interactions in particle physics.

Note the difference however, in plasma we deal with *real* particles (electrons) while in the case of nucleons, the screening is produced by *virtual* particles (π-mesons).

Anomalous nucleon magnetic moments

The picture that is implied by the Yukawa model is that a nucleon is continuously emitting and re-absorbing virtual π-mesons, i.e. each nucleon is surrounded by a cloud of virtual π-mesons.

Heisenberg uncertainty principle allows us to estimate spatial extent of the could: $\Delta E \Delta t \approx \hbar$. $\Delta E = M_\pi c^2 = 140$ MeV, therefore $\Delta t \approx \hbar / 140\text{MeV} = 4 \times 10^{-24}\, s$ (which not surprisingly is the correct order of magnitude for the strong interaction timescale). Using $R = c\Delta t$ we obtain 1 fm for the cloud radius.

P. Dirac has shown, based on his relativistic wave equation, that an electron should have a magnetic moment

$$\mu_e = -\frac{e\hbar}{2m_e c}.$$

Then according to Dirac theory nucleons (proton or neutron) should have following magnetic moments (because proton has charge $+e$ and neutron has charge of zero):

$$\mu_p = \frac{e\hbar}{2m_p c}, \qquad \mu_n = 0$$

It then came as great surprise that experimentally measured values were

$$\mu_p = 2.79\left(\frac{e\hbar}{2m_p c}\right), \qquad \mu_n = -1.91\left(\frac{e\hbar}{2m_p c}\right)$$

Further, it was then noticed that "excess" proton magnetic moment of 1.79 is about the same as the "deficiency" for neutron (-1.91).

In fact we can explain this discrepancy with the conjecture that proton exists as neutron $+\pi^+$ meson,

$$p \leftrightarrow n + \pi^+$$

while for neutron we have

$$n \leftrightarrow p + \pi^-$$

This is indeed corroborated by the scattering experiments!

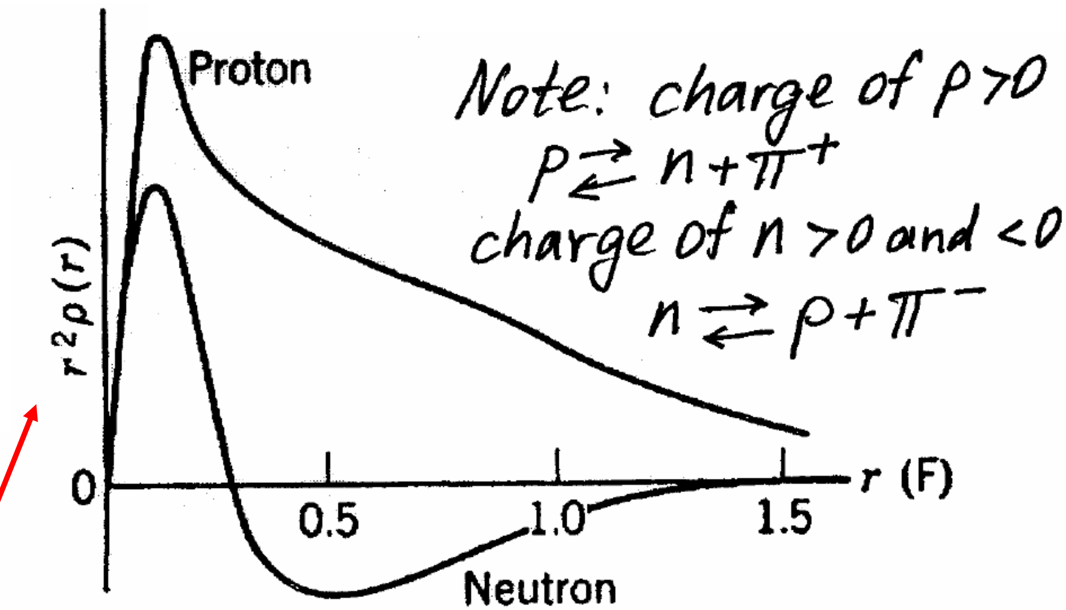


Fig. from Eisberg and Resnick, p.637, Fig.17-17)

Plasma oscillations – **collective effects**, example 2

Let us consider of quasi-neutral plasma in which ions are static and electrons freely moving. Let us look at an electron that has been displaced by distance x. Poisson equation for electric field,

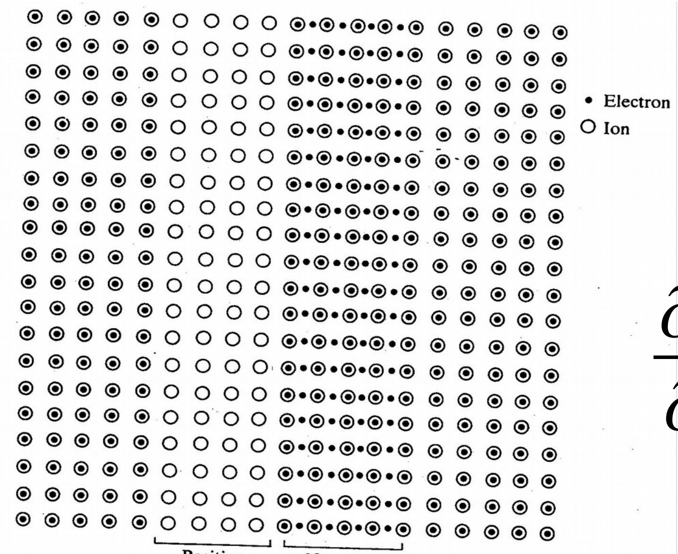


Fig. 1.1 Displacement of electrons in a cold plasma.

$$\nabla\cdot\vec{E}=\frac{\rho_e}{\varepsilon_0}$$ consider 1D case (E=E_x):

$$\frac{\partial E}{\partial x}\approx\frac{E}{x}=\frac{e\,n_0}{\varepsilon_0}\Rightarrow E=\frac{e\,n_0}{\varepsilon_0}x$$

and Newton's 2nd law give:

$$m_e\vec{a}=\vec{F}\Rightarrow m_e\ddot{x}=\text{-}eE=\frac{e^2 n_0}{\varepsilon_0}x\Rightarrow\ddot{x}=-\frac{e^2 n_0}{\varepsilon_0 m_e}x\equiv-\omega_{pe}^2 x\,;\Rightarrow\omega_p=\sqrt{\frac{n_0 e^2}{m\varepsilon_0}}$$

$$56.4\sqrt{n_0[m^{-3}]}$$

This describes oscillations at electron *plasma frequency*. Note it is only function of plasma number density. (for solar corona $\omega_p\approx 10^9$ Hz rad)

Landau damping – **collective effects**, example 3

Collective effects and the possibility of wave-particle interactions in plasma with Maxwellian distribution function (**DF**) give rise to a very peculiar from of wave damping which is **collisionless**!
(recall that damping is usually associated with dissipation or friction)

Key requirements:

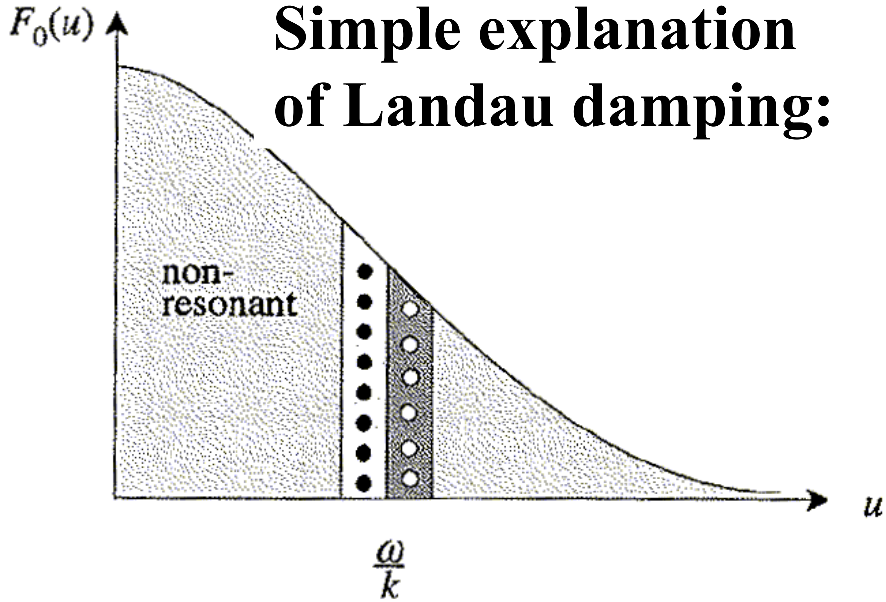
* Thermal spread of velocities in the distribution (i.e. Maxwellian)

* Wave has to have an electric field component along its propagation direction (remember magnetic field does no work on a physical system)

* for a harmonic wave $E(r,t)=\widetilde{E}(k,\omega)e^{i(kx-\omega t)}$ and with $\omega=\omega_r+i\gamma$ the growth/damping rate (imaginary part of the frequency) should satisfy $\gamma<<\omega_r$, otherwise wave does not exist (too damped).

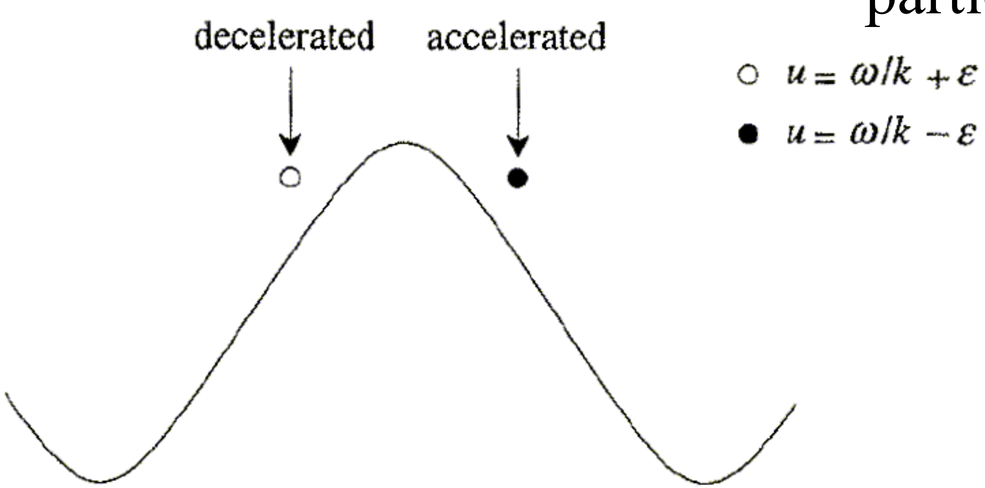
$Q=\omega_r/\gamma>>1$ is also know as the Quality or Q-factor of a wave.

Simple explanation of Landau damping:



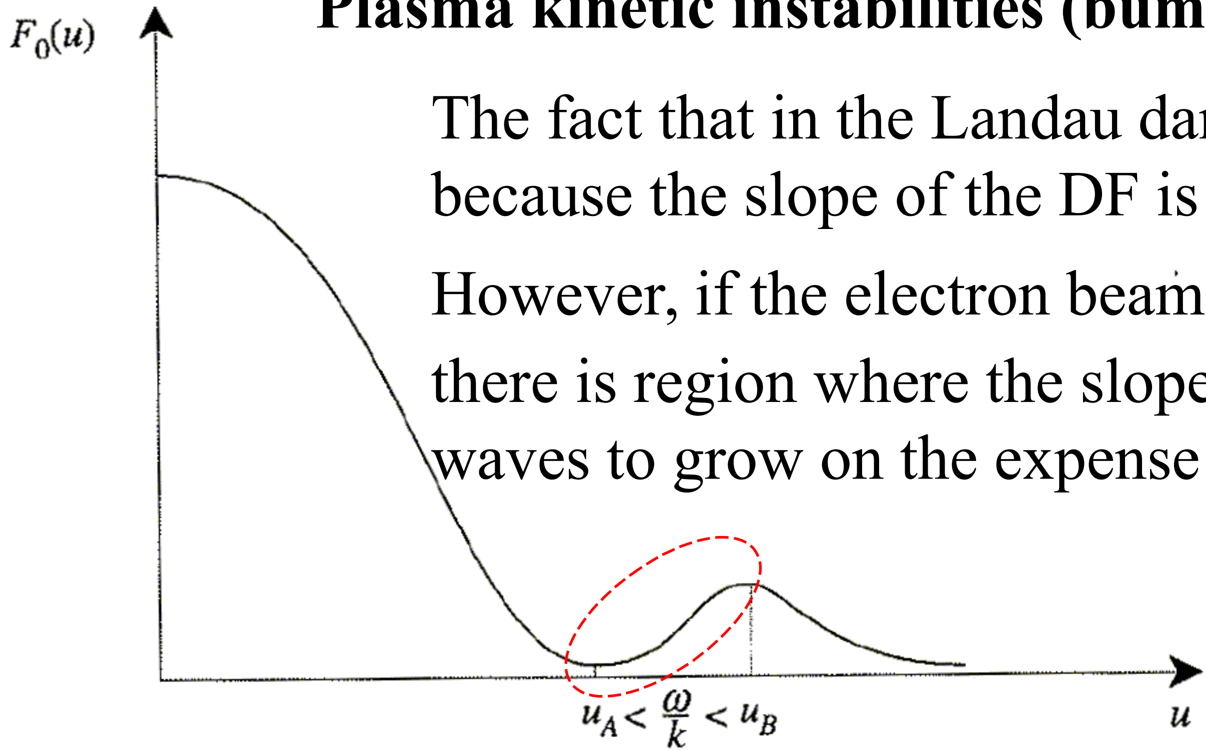
Particles which are faster than the wave give up their energy to the wave, while particles which are slower gain energy from the wave.

For a given phase speed of the wave, $u_{ph}=\omega/k$, in the Maxwellian distribution there are always more particles with $u< u_{ph}=\omega/k$ than particles with $u> u_{ph}=\omega/k$.



The net result of having Maxwellian distribution is that the wave gives up its momentum to the particles with $u \le u_{ph}$ and accelerates them. As a result the wave damps.

Plasma kinetic instabilities (bump-on-tail)



The fact that in the Landau damping wave damps is because the slope of the DF is <u>negative</u> everywhere.

However, if the electron beam ($u>>v_{th,e}$) is present then there is region where the slope is <u>positive</u>, which enables waves to grow on the expense of particle kinetic energy.

Mathematically, this is described by evaluating real and imaginary parts:

$$\omega_r \approx \omega_{pe}\left(1+(3/2)k^2\lambda_D^2\right)$$

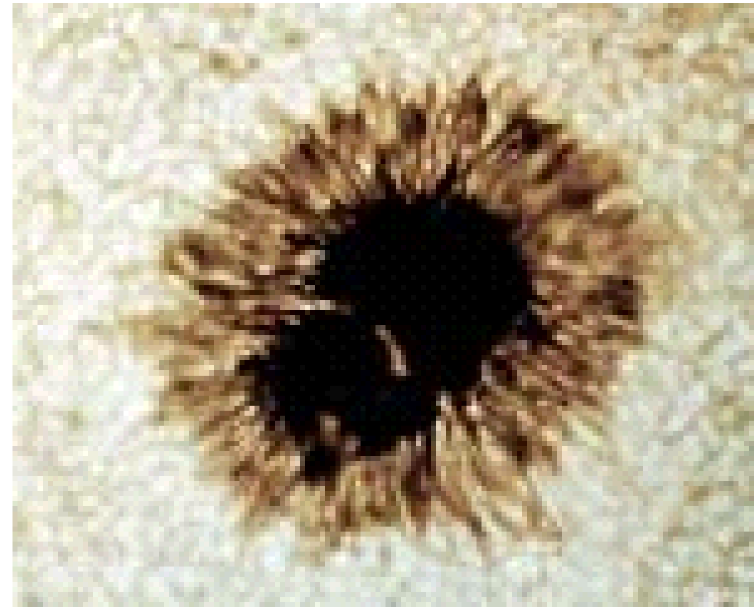
This is the dispersion relation for Langmuir waves.

$$\gamma = \frac{\pi\omega_{pe}^3}{2k^2}\frac{dF_0(u)}{du}\Big|_{u=\omega/k}$$

This is the growth rate for Langmuir waves. Note that when the slope is <u>positive</u> γ is positive thus the waves grow exponentially.

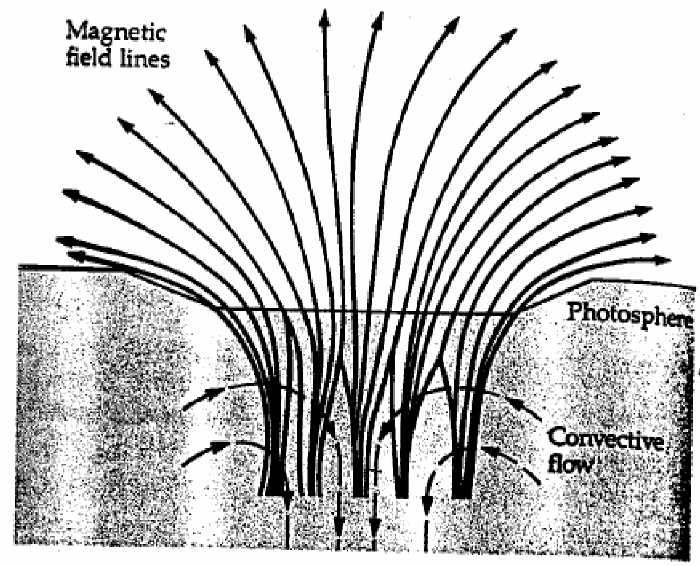
See Boyd & Sanderson book: *The physics of plasmas*, (2003). p.256

Applications of (space) plasma physics
Application 1 – Sunspots



Sunspots are dark spots on the surface of the Sun (at Photospheric level). They typically last for days However, large ones may exist for several weeks.

Sunspots are actually regions of strong magnetic field 0.1 Tesla. The magnetic field inhibits convection from below making sunspots cooler than surrounding plasma. Because of blackbody radiation law (Stefan-Boltzmann) $F = \sigma T^4$. sunspots appear as dark features.



A model of the magnetic fields in the photosphere that generate sunspots.

Let us apply plasma equilibrium conditions to create a simple sunspot model. On a large scale, bulk plasma dynamics is described by magnetohydrodynamics (MHD) – [fluid + magnetic field].

Plasma equation of motion can be written as Newton's 2 law

$$ma = F$$

Its MHD equivalent is:

$$\rho\left(\frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\vec{\nabla})\vec{V}\right) = -\vec{\nabla} p + \vec{j}\times\vec{B}$$

Where ρ is mass density, V is velocity, p is pressure, j is current density and B is the magnetic field.

In equilibrium time derivatives and velocities are zero. Thus we have:

$$0 = -\nabla p + j \times B$$

If we substitute $j = \nabla \times B / \mu_0$ where μ_0 is the permeability constant, we finally obtain (after simplification) the plasma equilibrium condition:

$$-\vec{\nabla}\left(p+\frac{B^2}{2\mu_0}\right)+\frac{1}{\mu_0}(\vec{B}\cdot\vec{\nabla})\vec{B}=0 \qquad (6)$$

Which is to say that *the gradient of the sum of thermal and magnetic pressures is equal to the magnetic tension.*

Because the field inside sunspot is constant, magnetic tension is zero:

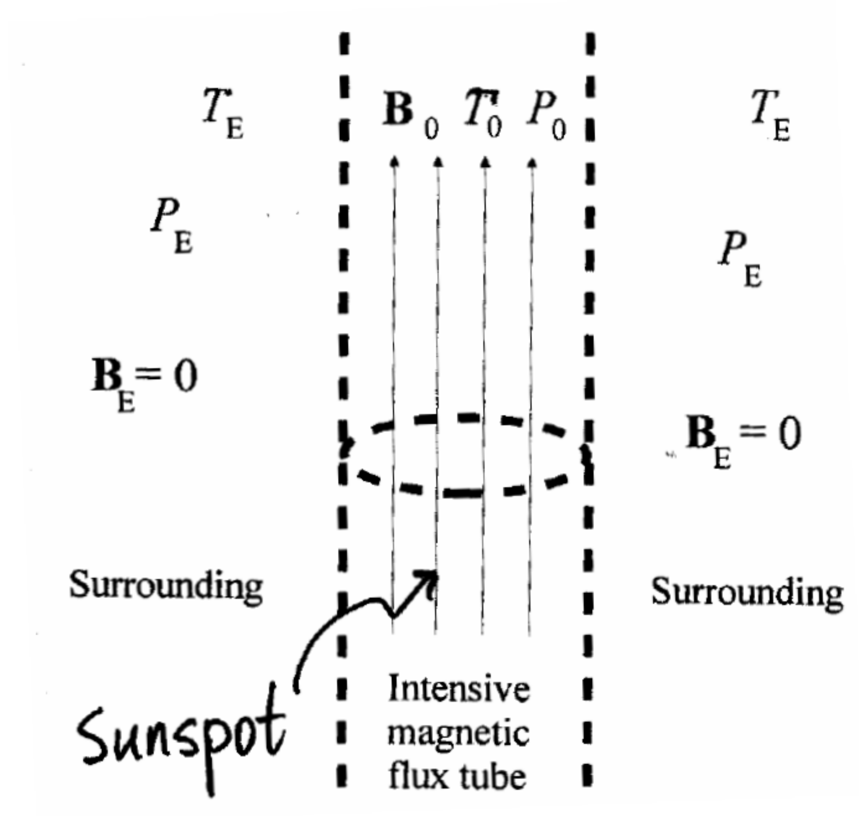
$$B_0 = const; \Rightarrow (B_0 \cdot \nabla) B_0 = 0$$

Thus, Eq.(6) reduces to

$$-\vec{\nabla}\left(p+\frac{B^2}{2\mu_0}\right)=0$$

which can be readily integrated to give

$$p+\frac{B^2}{2\mu_0}=const \qquad (7)$$



Rewriting the latter equation using sunspot sketch we obtain:

$$p_E = p_0 + \frac{B_0^{\ 2}}{2\mu_0}$$

Now use ideal gas law $p = n\,k\,T$, where n is number density (number of plasma particles per unit volume):

$$n_E k T_E = n_0 k T_0 + \frac{B_0^{\ 2}}{2\mu_0}$$

To a good approximation there is no density variation across the sun-spot i.e. $n_E = n_0$. Thus, rearranging we obtain:

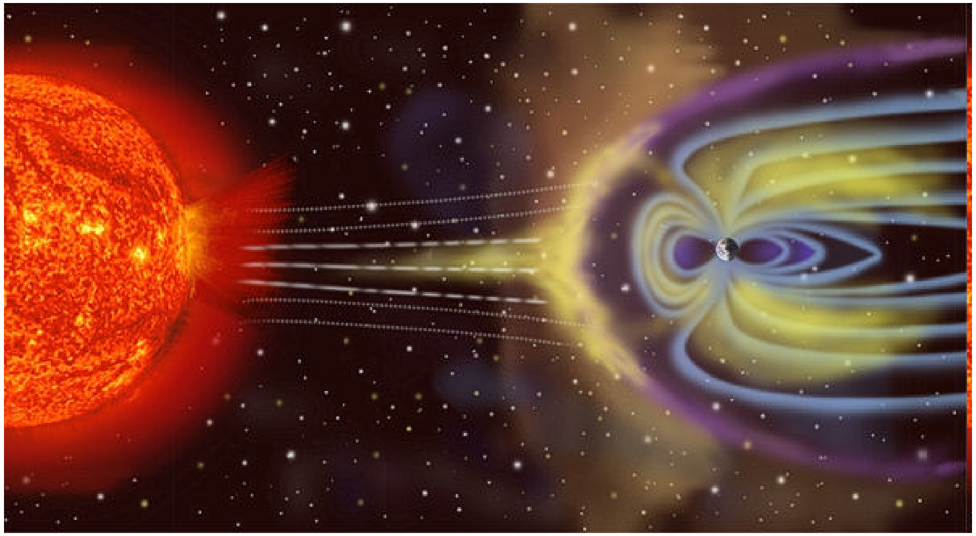
$$\frac{T_0}{T_E} = 1 - \frac{B_0^{\ 2}}{2\mu_0 (n_E k T_E)} = 1 - \frac{0.1^2}{2 \times 4\pi 10^{-7} (10^{23} 1.38 \times 10^{-23} 6000)} = 0.52$$

Thus finally,

$$T_0 = 0.52 T_E = 0.52 \times 6000 = 3100K$$

Then naturally, blackbody radiation law (Stefan-Boltzmann) $F = \sigma T^4$ readily explains a lesser emission flux from the sunspots.

Application 2 – Estimate of the earth magnetosphere size



Earth magnetosphere, which protects us from harmful energetic particles (cosmic rays) by trapping them via magnetic filed, is a consequence of balance between the thermal pressure of the solar wind and magnetic pressure of earth magnetic field.

Using Eq.(7): $p_{SW} = \dfrac{B_{EARTH}^{\;2}}{2\mu_0}$; $\quad nkT = \left(\dfrac{3\times 10^{-5}}{R^3}\right)^2 \dfrac{1}{2\mu_0}$;

Here we used ideal gas law and earth magnetic field simple model $B_{EARTH} = 3\times 10^{-5}/R^3$, where R is measured in earth radii.

$$R = \left(\frac{\left(3\times 10^{-5}\right)^2}{2\mu_0\left(nkT\right)}\right)^{1/6} = \left(\frac{\left(3\times 10^{-5}\right)^2}{2\times 4\pi 10^{-7}\left(4\times 10^6\times 1.38\times 10^{-23}\, 3\times 10^5\right)}\right)^{1/6} = 16.7$$

earth radii

Application 3 – calculation of spacecraft surface charging potential

The fact that electrons and ions have different thermal speeds (because of their different masses) gives rise to some interesting plasma phenomena such as Debye sheath formation, and turn, surface charging of conducting walls when they come in contact with plasma.

Characteristic speed of plasma particles of species α is called *thermal speed* and it is:

$$V_{th,\alpha} = \sqrt{\frac{kT}{m_\alpha}}$$

In this notation $\alpha = e$ for electrons and $\alpha = i$ for ions (a proton in the case of pure hydrogen plasma).

Because protons are 1836 times more massive than electrons, this implies that ions are moving $\sqrt{1836} \approx 43$ times slower.

Imagine a situation when a metal surface (a conductor) is brought in contact with plasma. Electrons will reach the surface first charging it up negatively. This means ions will be attached to the surface whist additional, on-coming electrons will be repelled.

Equilibrium is reached when electron flow (current) exactly balances ion current:

$$|J_e| = |-n_e e V_e| = J_i = n_i e V_i$$

This effect is called *Debye sheath* formation.

As electrons are more mobile than ions, in the equilibrium situation, we can assume that electrons are distributed according to the Boltzmann law:

$$n_e = n_0 \exp(-q_e \varphi / kT) = n_0 \exp(e\varphi / kT)$$

where φ is the negative potential of the wall we would like to calculate; while ion number density n_i is constant (ions move slowly, hence do not have time to react):

$$n_i = n_0 = const$$

The electron-ion current balance then gives

$$n_e e V_e = n_i e V_i,$$
$$n_0 \exp(e\varphi / kT) e V_{th,e} = n_0 e V_{th,i}$$

where electron and ion speeds where replaced by respective *thermal speeds*.

Simple algebra yields:

$$\exp(e\varphi / kT) = V_{th,i} / V_{th,e} = \sqrt{\frac{kT}{m_i}} / \sqrt{\frac{kT}{m_e}} = \sqrt{\frac{m_e}{m_i}}; \Rightarrow$$

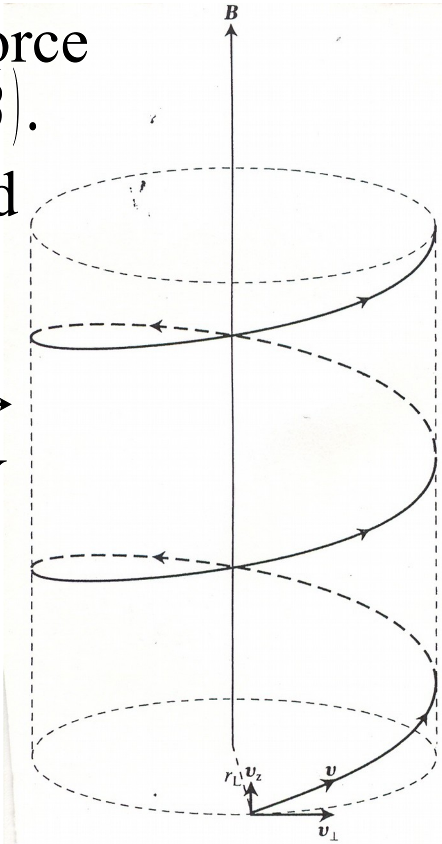
$$e\varphi / kT = \ln\sqrt{\frac{m_e}{m_i}} \Rightarrow \varphi = \frac{kT}{e}\ln\sqrt{\frac{m_e}{m_i}} = \frac{1.38\times10^{-23} 3\times10^{5}}{1.6\times10^{-19}}\ln\left(\sqrt{1/1836}\right) =$$

$$\varphi = -97.2 \approx -100 \text{ Volts!}$$

For example 100 V would be a potential that a satellite acquires (note we used $T = 3\text{x}10^5$ K solar wind temperature at earth distance from the sun) due to this plasma effect. Potential hazard – discharges!

Let's look at relation between some **characteristic frequencies**:
electron collision frequency ν_e, plasma frequency $f_p = \frac{\omega_p}{2\pi}$,
and *cyclotron frequency* $f_{ce} = \frac{\omega_{ce}}{2\pi} = \frac{1}{2\pi}\frac{eB}{m_e}$ for solar corona and fusion plasmas.

The Lorentz force
$\vec{F}_L = q\left(\vec{E} + \vec{V} \times \vec{B}\right).$
Case of B-field Only
$F_L = m_e a \Rightarrow$
$-e\vec{V} \times \vec{B} = m_e \dot{\vec{V}}$
Yields
$\dot{\vec{V}} = \omega_{ce} \times V$
Where
$\omega_{ce} = eB / m_e$
is cyclotron frequency.



Fusion plasmas n=10^{14}cm^{-3} T=10^{3}eV
Solar corona n=10^{9}cm^{-3} T=10^{2}eV
(note 1eV=11600K)

$$\nu_e = 2.9 \times 10^{-6} \ln \Lambda n T^{-3/2} \quad \text{Hz}$$
$$f_p = 8.98 \times 10^3 = 10^4 \sqrt{n} \quad \text{Hz}$$
$$f_{ce} = 2.8 \times 10^6 B \quad \text{Hz}$$

Here n [cm^{-3}]; T [eV]; B in [G] and Coulomb logarithm (for electrons) is

$$\ln(\Lambda) = \begin{matrix} 23 - \ln\left(n^{1/2} T^{-3/2}\right) \, for \; T < 10 \, eV \\ 24 - \ln\left(n^{1/2} T^{-1}\right) \, for \; T > 10 \, eV \end{matrix}$$

	f_{ce} Hz	f_p Hz	ν_e Hz
Fusion	2.8×10^{11}	3×10^{11}	1.4×10^{5}
Corona	2.8×10^{8}	3×10^{8}	53

Two important conclusions follow from these estimates:

1. For the both cases $f_{ce}/\nu_e \approx$ few $\times10^6$ which means that between every collision electrons rotate millions of times around magnetic field line. Thus for solar coronal and fusion plasma *magnetic field* plays far more important role as a restraining force than collisions.

2. For the both cases $f_{ce}/f_p \approx 1$. This coincidence is responsible for a great degree of complexity in the plasma behaviour.
A. Mathematically this makes the dispersion relations difficult to treat.
B. The Larmor radius is comparable to Debye radius:

$$\omega_{ce} = 2\pi f_{ce} = \frac{v_{th,e}}{r_L} \approx \omega_p = 2\pi f_{ce} = \frac{v_{th,e}}{\lambda_D} \Rightarrow r_L \approx \lambda_D$$

Charged particle dynamics

Equation of motion of a charged particle in B-field is

$$\dot{\vec{V}} = \left(e\vec{B}/m_e\right)\times\vec{V} = \vec{\omega}_{ce}\times\vec{V} \quad (1)$$

If we choose the direction of ω_{ce} along z-axis then we see that the acceleration $\vec{a}=\dot{V}$ has no z-component, which means that $V_z=const$.

It also follows from Eq.(1) that $\vec{V}\cdot\dot{V}=0$, because $\dot{V}\perp V$ and generally $(A\times B)\perp A\perp B$.

But $\vec{V}\cdot\dot{\vec{V}}=\dfrac{d}{dt}\left(\dfrac{\vec{V}^2}{2}\right)=0\Rightarrow\vec{V}^2=V^2=const$.

Physically this means that the magnetic field has no effect on the velocity component of the particle which is <u>parallel</u> to the field; and since the B-field induces a force that is always <u>perpendicular</u> to the direction of particle's motion, the particle energy is <u>conserved</u>.

Based on this a useful invariant can be introduced:

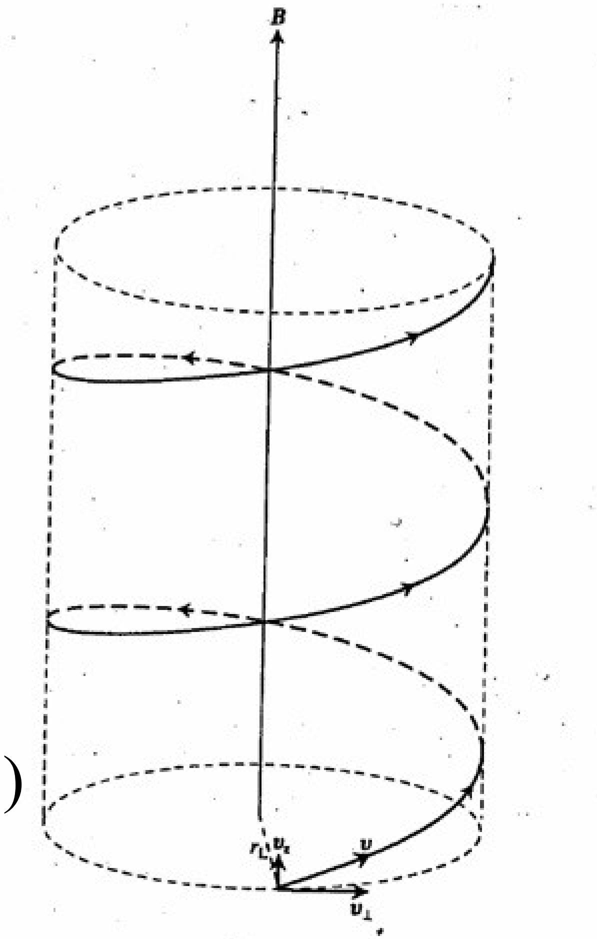
$$V_\perp=\sqrt{V^2-V_z^2}=\sqrt{V^2-V_\parallel^2}=const \qquad (2)$$

which is the magnitude of the velocity perpendicular to the B-field $V_\perp = V - V_z e_z$, depicted in this figure.

Writing Eq.(1) in the component form yields:

$$\dot{V}_x = \begin{vmatrix} e_x & e_y & e_z \\ 0 & 0 & \omega_{ce} \\ V_x & V_y & V_z \end{vmatrix}_x = 0 \cdot V_z - \omega_{ce} V_y = -\omega_{ce} V_y$$

$$\dot{V}_y = \begin{vmatrix} e_x & e_y & e_z \\ 0 & 0 & \omega_{ce} \\ V_x & V_y & V_z \end{vmatrix}_y = \omega_{ce} V_x - 0 \cdot V_z = \omega_{ce} V_x \qquad (3)$$



Helical path of an electron in a uniform magnetic field.

Fig. 1

Differentiating Eq.(3) with respect to time, we obtain

$$\ddot{V}_x = -\omega_{ce}\dot{V}_y = -\omega_{ce}^2 V_x$$

$$\ddot{V}_y = \omega_{ce}\dot{V}_x = -\omega_{ce}^2 V_y$$

The above equations describe what? Answer: simple harmonic motion

$$V_x = A_x \cos(\omega_{ce} t + \varphi); \qquad V_y = -A_y \sin(\omega_{ce} t + \varphi)$$

But $V_x^2 + V_y^2 = V_\perp^2 \Rightarrow A_x = A_y = V_\perp \Rightarrow$

$$V_x = V_\perp \cos(\omega_{ce} t + \varphi); \qquad V_y = -V_\perp \sin(\omega_{ce} t + \varphi) \qquad (4)$$

Eq.(4) can be integrated with respect to time:

$$x = x_0 + (V_\perp / \omega_{ce}) \sin(\omega_{ce} t + \varphi); \qquad (5.1)$$

$$y = y_0 + (V_\perp / \omega_{ce}) \cos(\omega_{ce} t + \varphi); \qquad (5.2)$$

$$z = z_0 + V_z t \qquad (5.3)$$

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Eq.(5.3) follows from integration of $V_z = const$.

Eq.(5.1-3) describe the dynamics of a charge (e.g. electron) in a uniform magnetic field in full. Here (x_0, y_0) are constants of integration and they are coordinates of the centre of a circular motion, which is part of the charge's helical path shown in Fig.1.

The amplitude V_{perp}/ω_{ce} in Eqs.(5.1-2) is known as the Larmor radius, which was mentioned before, when discussing magnetised plasmas.

Thus we can specify V_{th} in the formula $\rho_e = V_{th}/\omega_{ce}$ as $V_\perp$, i.e. perpendicular component of the electron's thermal speed.

Hence we have

$$\rho_e = \frac{V_{perp}}{\omega_{ce}} = 7.5\times 10^{-5}\left(\frac{V_{perp}}{1.3\times 10^{7}\,\mathrm{m/s}}\right)\left(\frac{B}{1\ \mathrm{Tesla}}\right)^{-1}\mathrm{m} \qquad (6)$$

The dependence of ρ_e on $V_\perp$ and B follows from the fact that the magnetic field provides the centripetal force which holds otherwise free electron in circular perpendicular motion.

Therefore ρ_e is the distance at which the centripetal force balances the Lorentz force:

$$ma = m\frac{V_\perp^2}{\rho_e} = eV_\perp B \Rightarrow$$

$$\frac{V_\perp}{\rho_e} = \frac{eB}{m} = \omega_{ce} \Rightarrow \rho_e = \frac{V_\perp}{\omega_{ce}}$$

which is identical to Eq.(6).

Eq.(6) indicates that in weak magnetic field, the force causing the electron to deviate from straight-line motion is small, and hence ρ_e is large.

Similar effect on the orbit is produced by electrons being too energetic, i.e. hot plasmas.

Guiding centre drift approximation:

It is convenient to write the electron position as

$$\vec{x} = \vec{x}_c + \frac{\vec{V} \times \vec{\omega}_{ce}}{\omega_{ce}^2} \qquad (8)$$

which implicitly defines the guiding centre position – see Fig.2

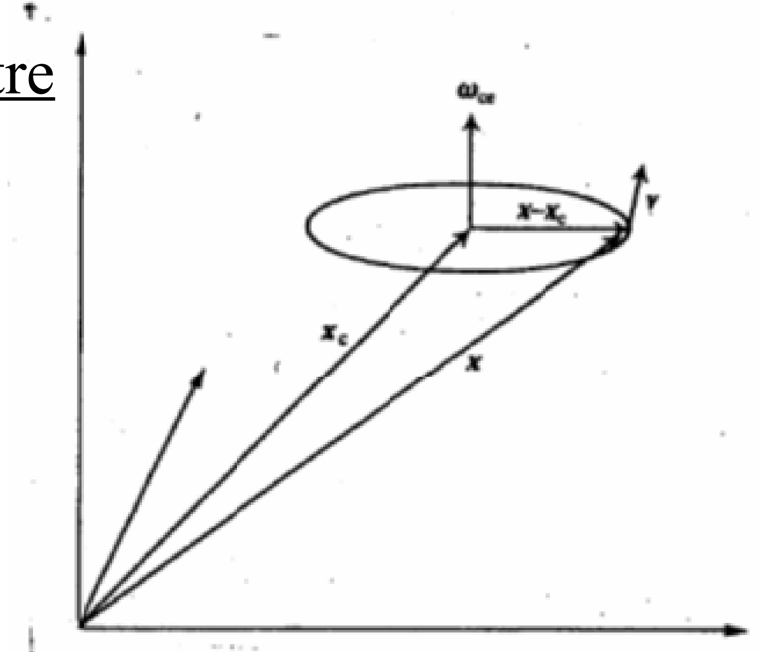
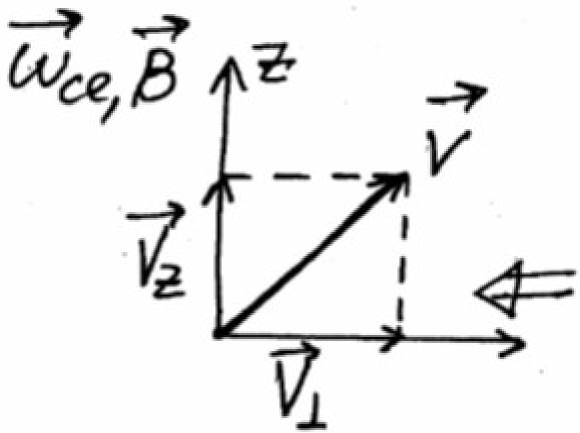


Fig. 2 Guiding centre position x_c.

Clearly the 2nd term in Eq.(8) describes dynamics perpendicular to the magnetic field (because $\vec{\omega}_{ce} \| \vec{B}$).



i.e. $\vec{V} \times \vec{\omega}_{ce}$ would be pointing out of the plane of the figure, shown on the left.

Let us show however that the 1st term in Eq.(8) indeed describes the parallel dynamics, i.e. that of the guiding centre. For this we have to differentiate Eq.(8) w.r.t. time and use Eq.(1):

$$\dot{\vec{x}} = \dot{\vec{x}}_c + \frac{\dot{\vec{V}} \times \vec{\omega}_{ce}}{\omega_{ce}^2} = \dot{\vec{x}}_c + \frac{\left(\vec{\omega}_{ce} \times \vec{V}\right) \times \vec{\omega}_{ce}}{\omega_{ce}^2}$$

Then vector identity states:

$$\vec{A} \times \left(\vec{B} \times \vec{C}\right) = \left(\vec{C} \times \vec{B}\right) \times \vec{A} = \vec{B}\left(\vec{A} \cdot \vec{C}\right) - \vec{C}\left(\vec{A} \cdot \vec{B}\right)$$

Thus we obtain:

$$\dot{\vec{x}} = \dot{\vec{x}}_c + \frac{\vec{V}\omega_{ce}^2 - \vec{\omega}_{ce}\left(\vec{\omega}_{ce} \cdot \vec{V}\right)}{\omega_{ce}^2} \qquad (9)$$

Note however that $\dfrac{\vec{\omega}_{ce}\left(\vec{\omega}_{ce}\cdot\vec{V}\right)}{\omega_{ce}^{2}}$ is nothing but V_z, i.e. electron velocity component parallel to B-field.

Therefore, Eq.(9) can be written as

$$\dot{\vec{x}} = \dot{\vec{x}}_c + \vec{V} - \vec{V}_z$$

but because $\dot{\vec{x}} = \vec{V}$, $\dot{\vec{x}}_c - \vec{V}_z$ should be zero:

$$\dot{\vec{x}}_c - \vec{V}_z = 0 \Rightarrow \dot{\vec{x}}_c = \vec{V}_z$$

Note also that

$$\left|\vec{x} - \vec{x}_c\right| = \rho_e$$

Now x_c is the mean position of the electron if the rapid variations (rotation) with frequency ω_{ce} is averaged out. This means that position x_c and also $\dot{\vec{x}}_c = \vec{V}_z$ (i.e. drift velocity if the guiding centre) often contains ALL the information required.

By calculating x_c (and hence $\dot{\vec{x}}_c = \vec{V}_d$, we can follow the path of the electron over a timescale long compared to ω_{ce}^{-1}.

In general when $B = B(r, t)$ i.e. magnetic field <u>changes</u> in space and time, particle dynamics is rather complicated.

However, when these changes are
(i) large-scale, i.e. $L_{variation} >> \rho_e$ and
(ii) slow, i.e. $T_{variation} >> \omega_{ce}^{-1}$,

where $L_{variation}$ and $T_{variation}$ are spatial and temporal scales of variation of the magnetic field, then the guiding centre approximation applies.

In this approach the helical trajectory of a particle in magnetic field is approximated by a smooth drift motion of the guiding centre as depicted in Fig. 4.

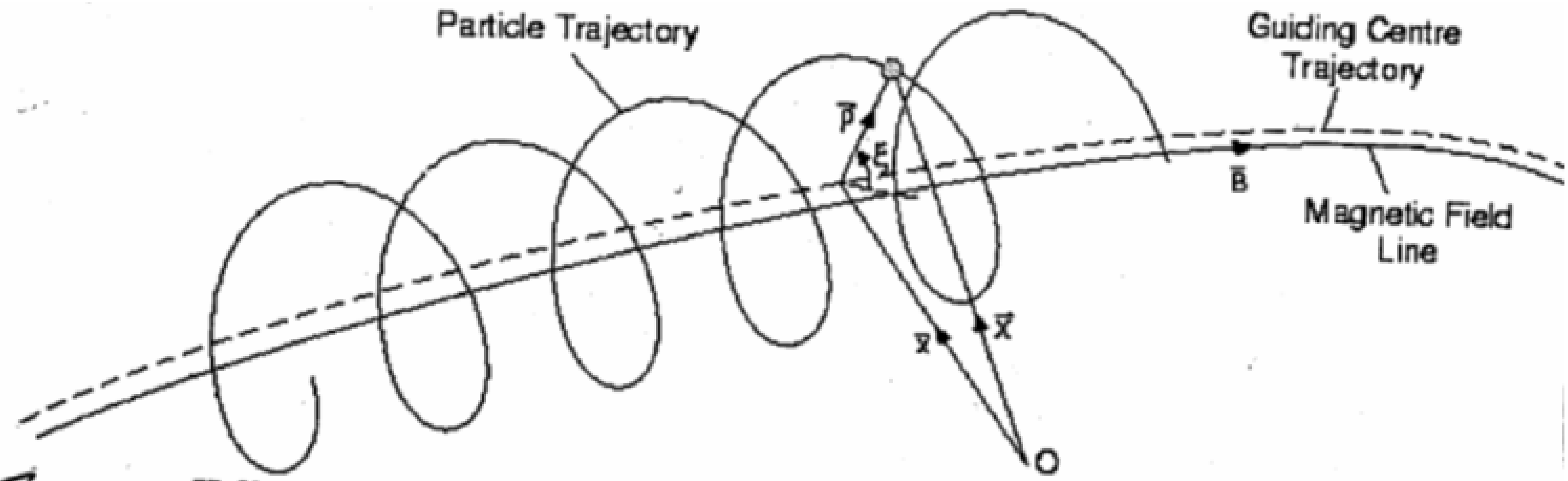


Fig. 4 Helical trajectory of a charged particle around a field line with guiding centre trajectory indicated.

Let electron be subjected to an impulsive collision , which by definition leaves its position unchanged (but NOT that of a guiding centre!), but instantaneously changes V to $V+\Delta V$.

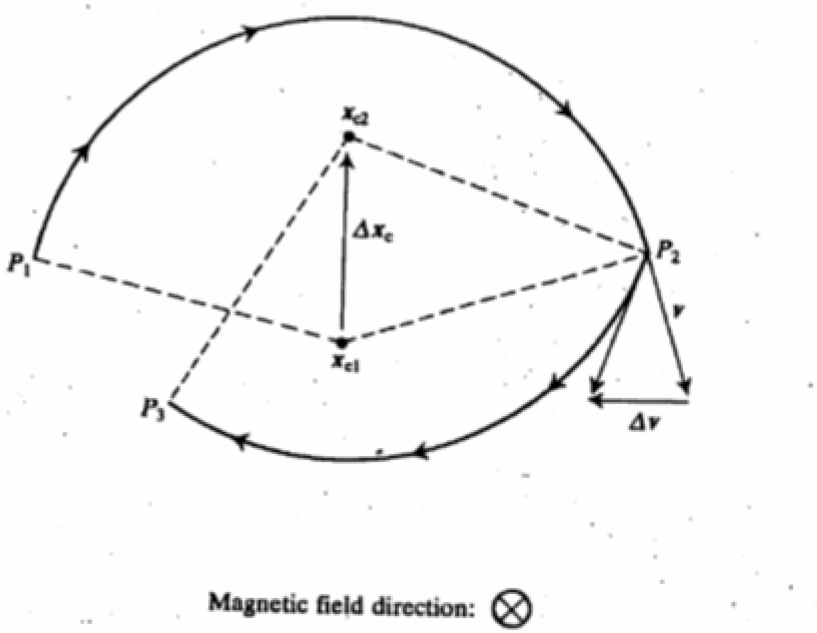
Hence,

$$\vec{x}=\vec{x}_{c1}+\frac{\vec{V}\times\vec{\omega}_{ce}}{\omega_{ce}^{2}}=\vec{x}_{c2}+\frac{\left(\vec{V}+\Delta\vec{V}\right)\times\vec{\omega}_{ce}}{\omega_{ce}^{2}}\Rightarrow$$

$$\Delta\vec{x}_{c}=\vec{x}_{c2}-\vec{x}_{c1}=-\frac{\Delta\vec{V}\times\vec{\omega}_{ce}}{\omega_{ce}^{2}}=\frac{\vec{\omega}_{ce}\times\Delta\vec{V}}{\omega_{ce}^{2}}.$$

$$\text{Note that } \vec{A}\times\vec{B}=-\vec{B}\times\vec{A}$$

The instantaneous step Δx_c in guiding centre position in perpendicular direction to the magnetic field is proportional to the <u>momentum transfer</u> ΔV. This shown in the next figure:



Instantaneous guiding centre step Δx_c due to impulsive collision at P_2.

Let us consider force F per unit mass. Hence in Eq.(1) extra term F appears:

$$\dot{\vec{V}} = \vec{\omega}_{ce} \times \vec{V} + \vec{F} \qquad (10)$$

Differentiating Eq.(8) and using Eq.(10) gives:

$$\dot{\vec{x}} = \dot{\vec{x}}_c + \frac{\dot{\vec{V}} \times \vec{\omega}_{ce}}{\omega_{ce}^2} = \dot{\vec{x}}_c + \frac{\left(\vec{\omega}_{ce} \times \vec{V}\right) \times \vec{\omega}_{ce}}{\omega_{ce}^2} + \frac{\vec{F} \times \vec{\omega}_{ce}}{\omega_{ce}^2}.$$

Recall from Eq.(9) that

$$\frac{\left(\vec{\omega}_{ce} \times \vec{V}\right) \times \vec{\omega}_{ce}}{\omega_{ce}^2} = \vec{V} - \vec{V}_z$$

Hence:

$$\cancel{\dot{\vec{x}}} = \dot{\vec{x}}_c + \cancel{\vec{V}} - \vec{V}_z + \frac{\vec{F} \times \vec{\omega}_{ce}}{\omega_{ce}^2} \Rightarrow$$

$$\dot{\vec{x}}_c = \vec{V}_z - \frac{\vec{F} \times \vec{\omega}_{ce}}{\omega_{ce}^2} = V_z \vec{e}_z + \frac{\vec{\omega}_{ce} \times \vec{F}}{\omega_{ce}^2}. \qquad (11)$$

The second term on the right hand side is the guiding centre drift velocity, V_d. It is perpendicular to both (i) the applied force F and magnetic field B ($\omega_{ce} \parallel B$). Using an electric field as an example:

$$\vec{F} = -\frac{e\vec{E}}{m} \Rightarrow \vec{V}_d = -\frac{e}{m}\frac{\vec{\omega}_{ce} \times \vec{E}}{\omega_{ce}^2}$$

$$-\frac{e}{m}\left(\frac{m^2}{e^2 B^2}\right)\left(\frac{e\vec{B} \times \vec{E}}{m}\right) = -\frac{\vec{B} \times \vec{E}}{B^2} = \frac{\vec{E} \times \vec{B}}{B^2}$$

$$\vec{V}_d = \frac{\vec{E} \times \vec{B}}{B^2} \qquad (12)$$

This is a typical example of particle motion in plasma:
(i) rapid rotation in the plane perpendicular to the magnetic field with frequency ω_{ce} and radius ρ_e;
(ii) parallel to the magnetic field motion with constant velocity V_z.;
(iii) slow drift perpendicular both to the applied force and the magnetic field with V_d given by Eq.(12).

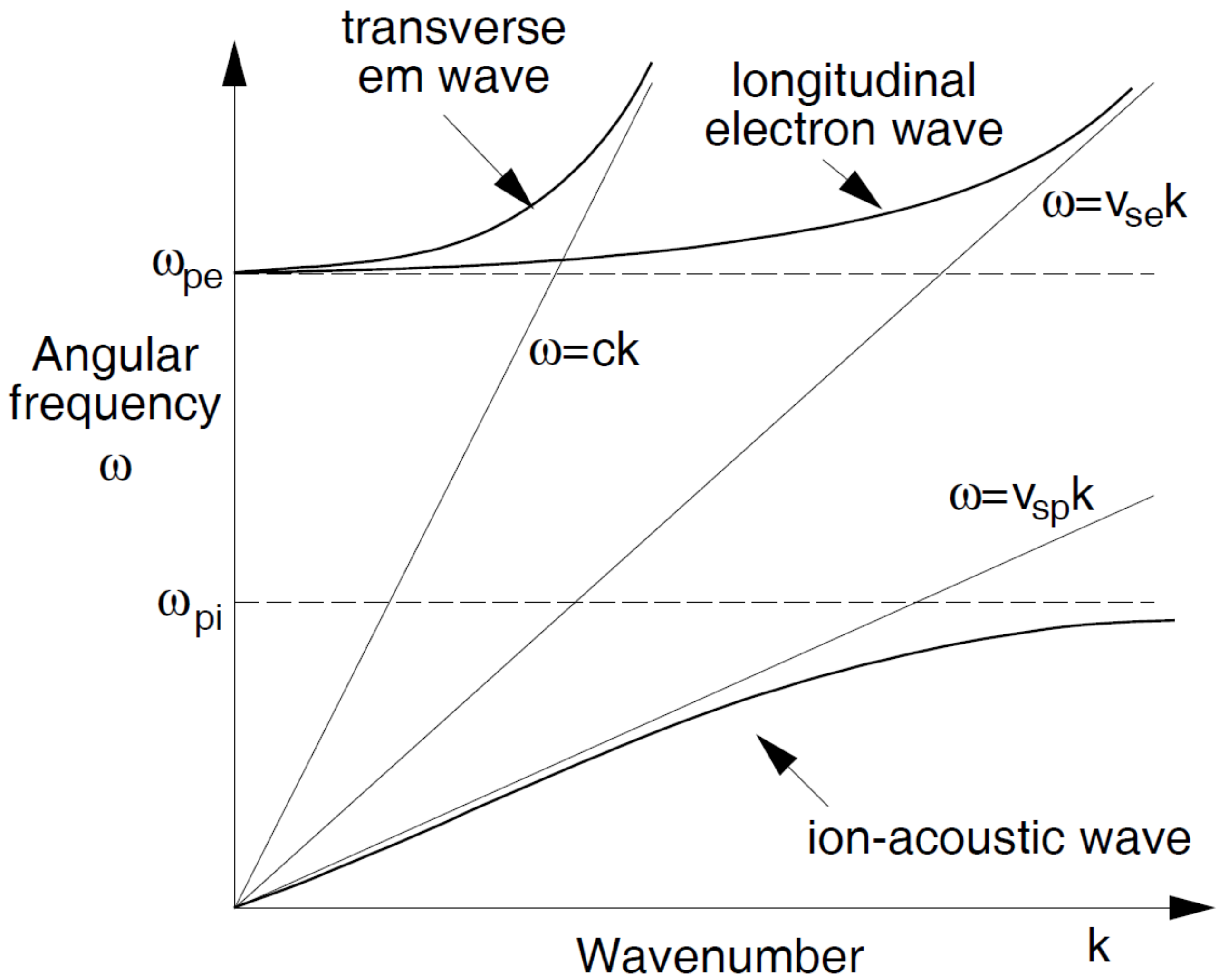
Expression given in Eq.(12) is called ExB drift.
There are also other types of drifts known:
1) drift due to gradients in the back ground magnetic fields ∇B ;
2) curvature drift; the one associated with the curvature of B;
3) polarization drift; the one associated with time varying electric fields;

4) gravity drift; when gravity plays the role of an external force F, etc
These are beyond the scope of this lecture -- details in
Chap. 2, *Plasma Dynamics*, by R. O. Dendy,
Chap. 2, *The Physics of Plasmas*, by T. J. M. Boyd, J. J. Sanderson

Next three slides should give a broad idea what types of waves exist in unmagnetised / magnetised plasmas

See SELF-STUDY MATERIAL based on "Plasma Dynamics" by R. O. Dendy at the end, starting on slide 60

B=0

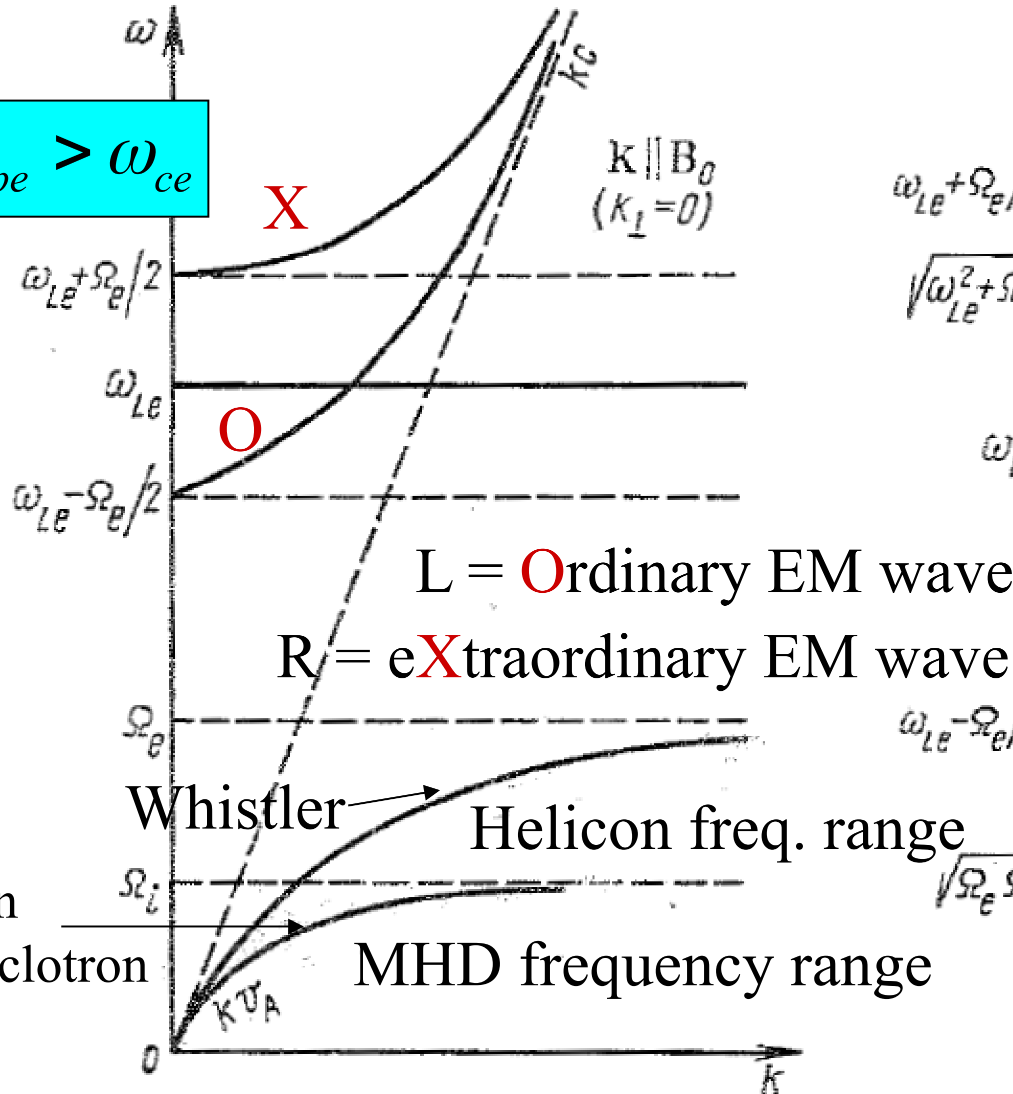


Dispersion relations for the three wave modes supported in an isotropic (unmagnetized) warm plasma.

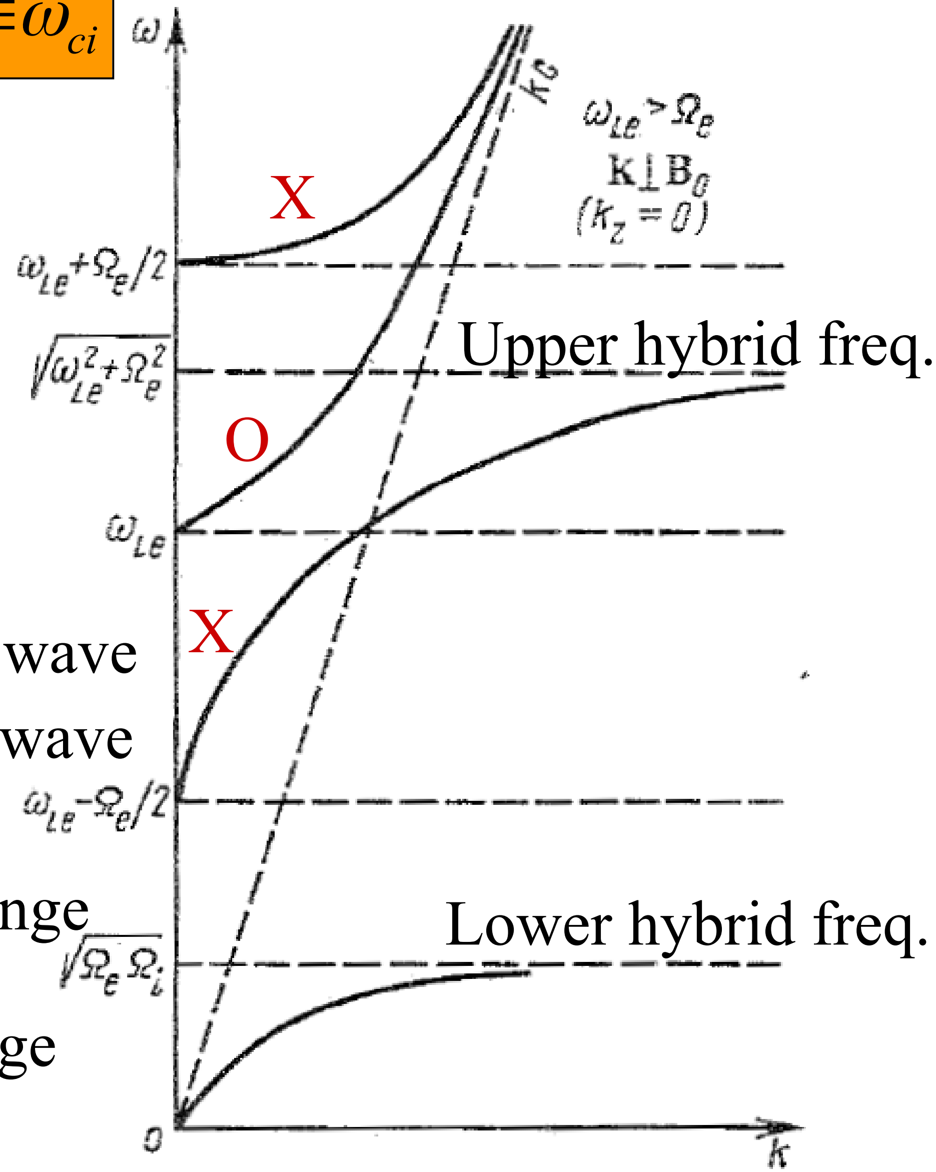
B≠0

$$\omega_{Le} \equiv \omega_{pe}; \Omega_e \equiv \omega_{ce}; \Omega_i \equiv \omega_{ci}$$

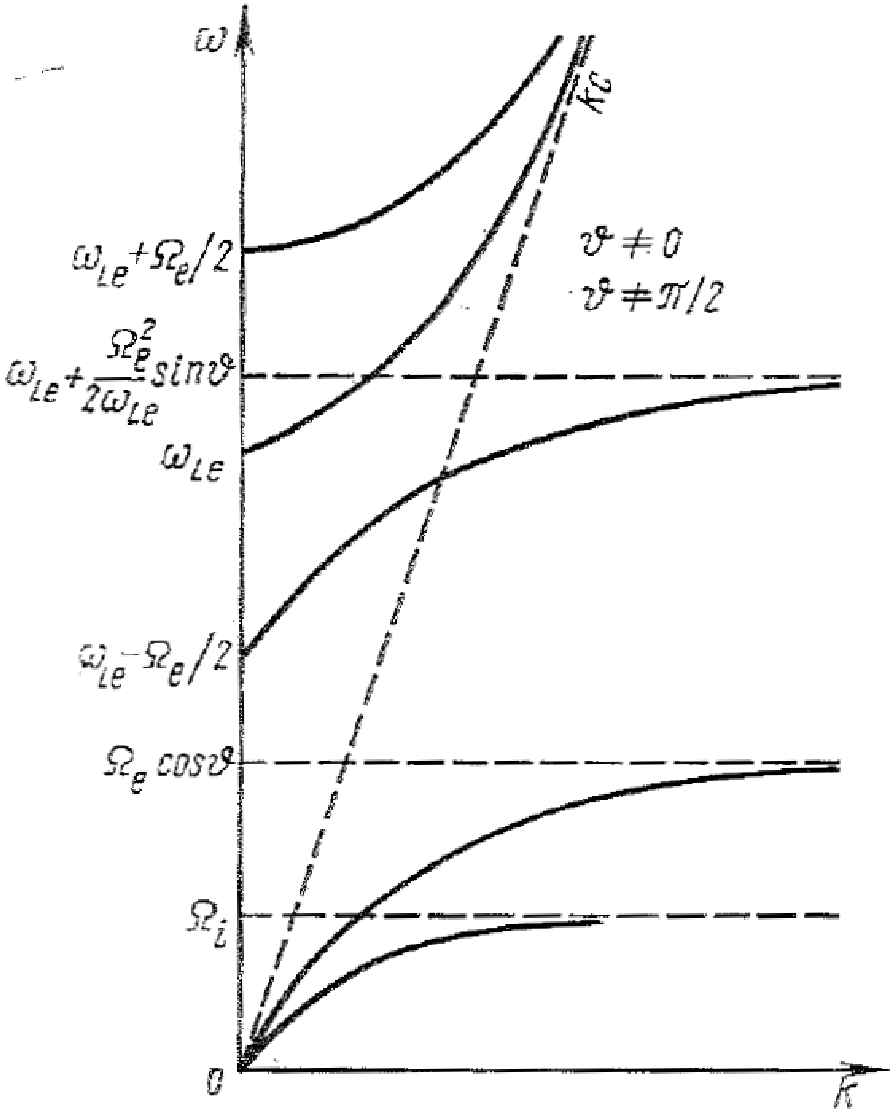
$$\omega_{pe} > \omega_{ce}$$



Spectra of waves for parallel propagation.

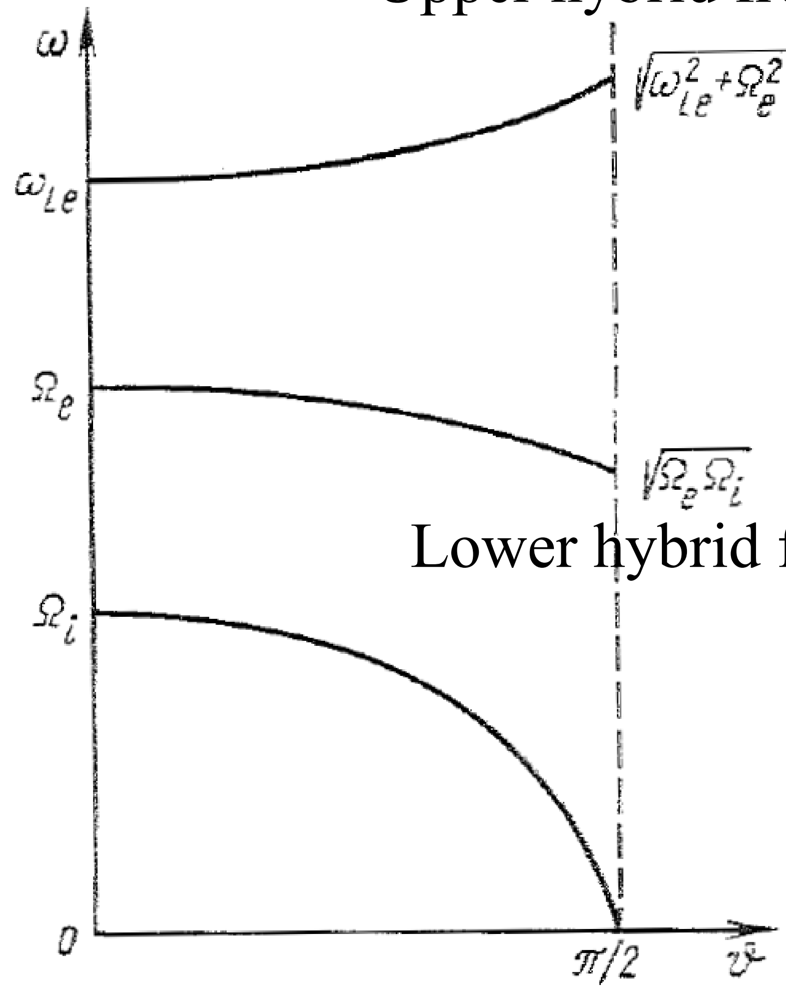


Perpendicular propagation



Spectra of waves arbit. θ propagation.

Upper hybrid freq.



Lower hybrid freq.

Spectra of longitudinal waves

Elements of plasma kinetics: Generalized Ohm's law

this can be readily obtained from the electron's equation of motion:

$$m_e\left(\frac{\partial \vec{v}_e}{\partial t}+(\vec{v}_e\cdot\nabla)\vec{v}_e\right)=-e\vec{E}-e\vec{v}_e\times\vec{B}-\frac{1}{n}\nabla\cdot P_e-m_e\nu_{ei}(\vec{v}_e-\vec{v}) \quad (4.1)$$

v is bulk ion velocity, P_e is electron pressure tensor, ν_{ei} electron-ion collision frequency.

If we put $v_e = v - j/ne$ and neglect terms of the order m_e/m_i, then the generalized Ohm's law is readily obtained:

$$\boxed{\vec{E}+\vec{v}\times\vec{B}-\eta\vec{j}}-\boxed{\frac{\vec{j}\times\vec{B}}{ne}}+\boxed{\frac{1}{ne}\nabla\cdot P_e}-\boxed{\frac{m_e}{ne^2}\left(\frac{\partial\vec{j}}{\partial t}+\nabla\cdot\left(\vec{j}\vec{v}+\vec{v}\vec{j}-\frac{\vec{j}\vec{j}}{ne}\right)\right)}=0 \quad (4.2)$$

This is usually used to determine, <u>which term is responsible</u> for the providing the <u>electric field</u>, be it reconnection or kinetic effects in waves. (First three terms give MHD Generalized Ohm's law.)

Important kinetic scales (on an example of reconnection inflow-outflow):

c/ω_{pi} ion inertial length -- Hall term

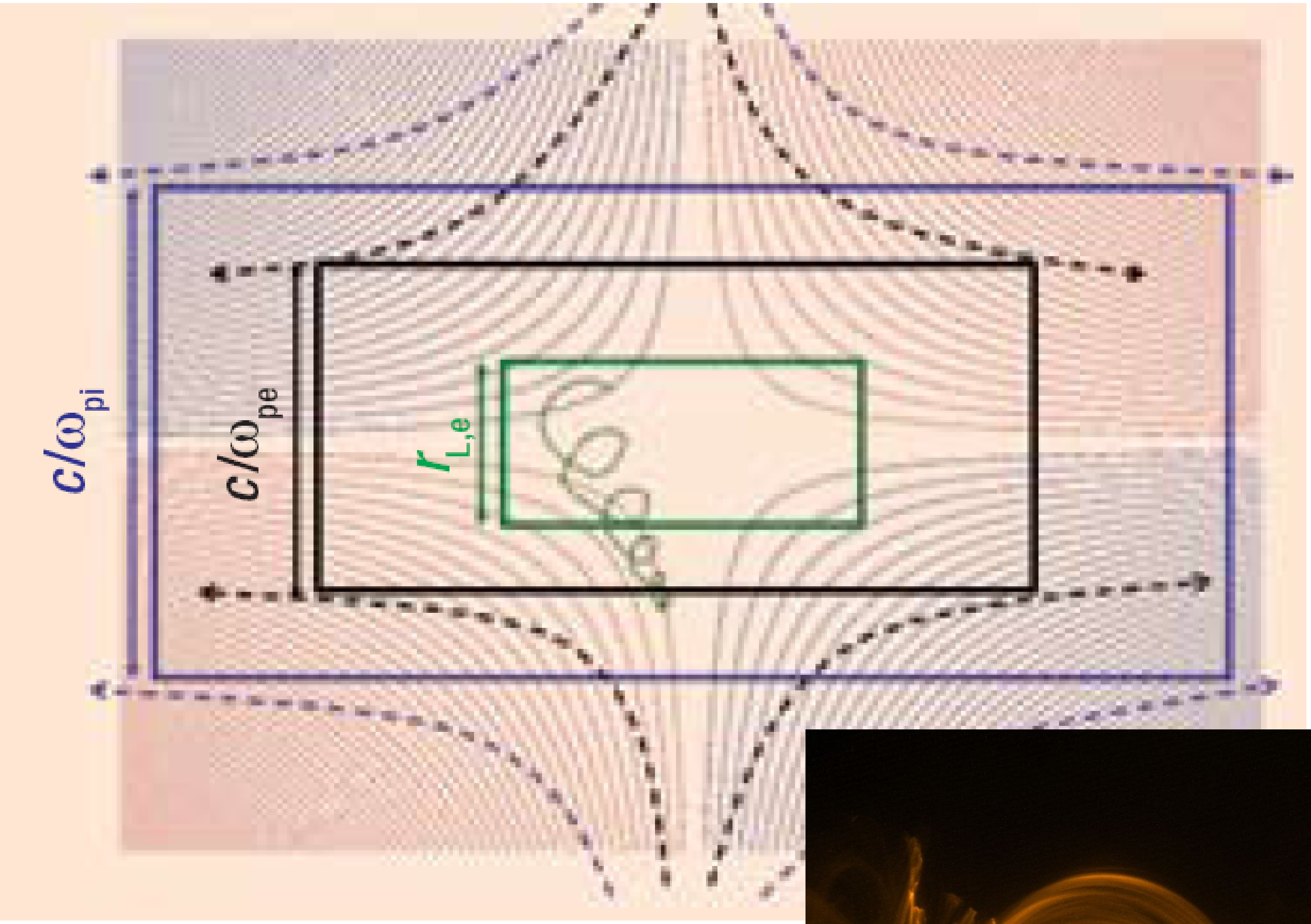
$c/\omega_{pi} \approx 10$ m

c/ω_{pe} electron inertial length -- electron inertia term

$c/\omega_{pe} \approx 10^{-1}$ m

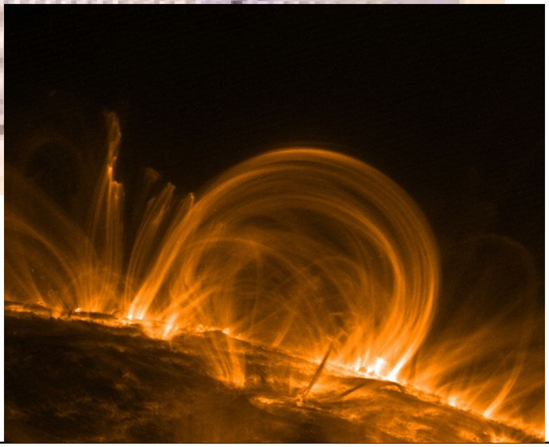
$r_{L,e}$ electron Larmor radius -- electron pressure tensor

$r_{L,e} \approx 10^{-3}$ m



MHD scale $\approx$ few 10^7 m!

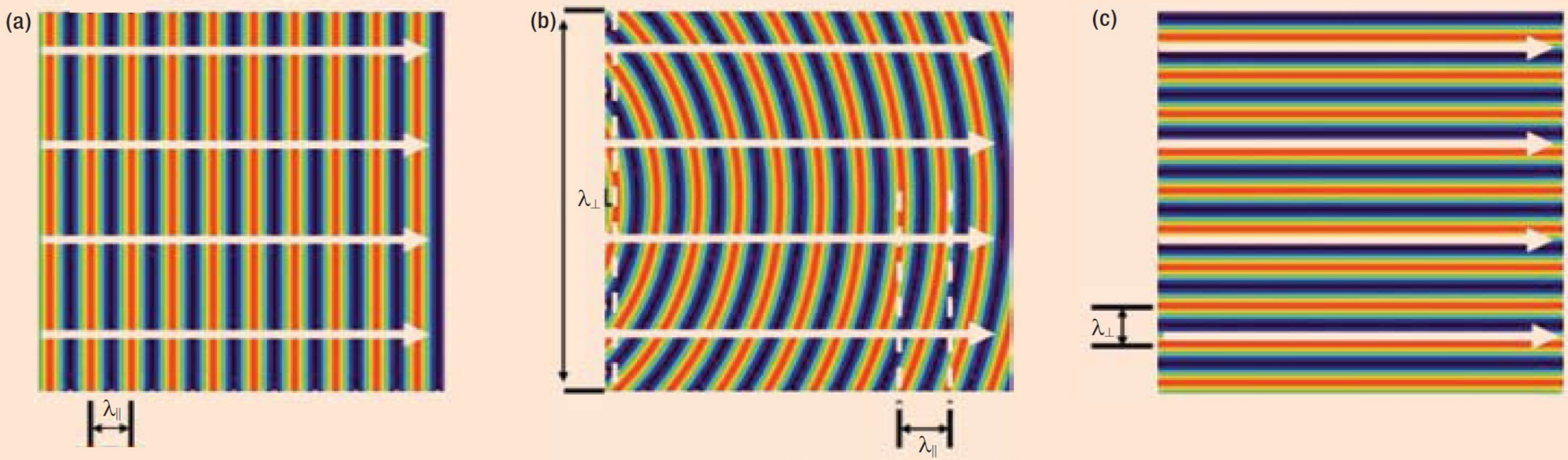
Tsiklauri, Astron & Geophys, 50, pp. 5.32 (2009)



Kinetic effects in waves

Alfven wave (AW) is a low-frequency electromagnetic (transverse) wave in a conducting, magnetised fluid. Two key ingredients of any wave are (i) restoring force and (ii) inertia. In the case AW these are (i) background magnetic field tension and (ii) inertia of ions.

AW dispersion relation $\omega = k_{\parallel} v_A$, where $v_A = B_0 / \sqrt{\mu_0 \rho}$ is the Alfven speed, $k_{\parallel}$ is the parallel to B_0 wavenumber.



4: A sketch of parallel, at an angle, and perpendicular propagation of a wave.
(a) Parallel: $\lambda_{\perp} = 1/k_{\perp}$ is ∞; $\lambda_{\parallel} = 1/k_{\parallel}$ is finite. (b) Angle: $\lambda_{\perp} = 1/k_{\perp}$ is finite; $\lambda_{\parallel} = 1/k_{\parallel}$ is finite. (c) Perpendicular: $\lambda_{\perp} = 1/k_{\perp}$ is finite; $\lambda_{\parallel} = 1/k_{\parallel}$ is ∞.

In MHD approximation AW has $E_\perp \neq 0$ and $E_\parallel = 0$. In terms of k's: $k_\perp = 0$ and $k_\parallel \neq 0$.

In full kinetic approach, if $\lambda_\perp = 2\pi/k_\perp$ of AW approaches the small kinetic scales such as $r_{L,i} = v_{th,i}/\omega_{pi}$ or c/ω_{pe}, $E_\parallel \neq 0$. This can have serious consequence for particle acceleration. Such waves are called *dispersive Alfven waves* (DAW).

Properties of DAWs can be quantified using collisionless (i.e. without dissipation) two-fluid theory α = e,i:

$$\frac{\partial \vec{v}_\alpha}{\partial t} + (\vec{v}_\alpha \cdot \nabla)\vec{v}_\alpha = \frac{q_\alpha}{m_\alpha}(\vec{E} + \vec{v}_\alpha \times \vec{B}) - \frac{1}{m_\alpha n_\alpha} \nabla \cdot P_e \qquad (4.3)$$

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \vec{v}_\alpha) = 0 \qquad (4.4)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}; \qquad \nabla \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

which are equations of motion of species alpha, continuity equations, and relevant Maxwell equations.

It is convenient to introduce usual scalar, φ, and vector, $\boldsymbol{A}$, potentials. Note that due to $\beta = P_{thermal}/P_{magnetic} = nkT/[B^2/(2\mu_0)] << 1$ approximation, perpendicular component of $\boldsymbol{A}$ can be ignored. Then,

$$E = -\nabla\varphi - \frac{\partial A_z}{\partial t}\hat{z} \qquad (4.5)$$

we assume that $\mathbf{B}_0 \| OZ$. Then from Eq.(4.5) it follows that

$$E_{\|} = E_z = -\frac{\partial\varphi}{\partial z} - \frac{\partial A_z}{\partial t} \qquad (4.6)$$

$$E_{\perp} = -\nabla_{\perp}\varphi \qquad (4.7)$$

Total current due to the AW is then

$$J = \mu_0^{-1}(\nabla\times\nabla\times(A_z\hat{z})) = \nabla\nabla\cdot(A_z\hat{z}) - \nabla^2(A_z\hat{z}) \qquad (4.8)$$

$$J_{\|} = -\mu_0^{-1}\nabla_{\perp}^{2}A_z \qquad (4.9)$$

$$J_{\perp} = \mu_0^{-1}\nabla_{\perp}\frac{\partial A_z}{\partial z} \qquad (4.10)$$

Inertial Alfven Waves (IAWs)

DAWs are subdivided into IAWs or Kinetic Alfven Waves (KAWs) depending on the relation $\beta < m_e/m_i$, i.e. $v_A > v_{th,i}, v_{th,e}$

when $\beta << m_e/m_i$ dominant mechanism for sustaining $E_{||}$ is parallel electron inertia (that is why such waves are called *inertial* AW).

Thus in Eq.(4.3) for electrons we ignore pressure term $O(v_{th,e}{}^2)$ compared to the inertia term:

$$\frac{\partial v_{e||}}{\partial t} = \frac{q_e}{m_e} E_{||} \qquad (4.11)$$

We can substitute $E_{||}$ from Eq.(4.6), then eliminate $\partial v_{e||} / \partial t$ term using Eq.(4.9) with $\partial / \partial t$ applied beforehand, (note that $J_{||} = - env_{e||}$) and also using Eq.(4.4) with O(3) (third order) terms neglected (i.e. $-\ \mu_0 e v_{e||} \nabla \cdot (n v_{e||})$ to be specific). This readily yields:

$$\left(\lambda_e^2 \nabla_\perp^2 - 1\right) \frac{\partial A_z}{\partial t} = \frac{\partial \varphi}{\partial z} \qquad (4.12) \quad \text{where} \quad \lambda_e = \frac{c}{\omega_{pe}}$$

In the $\omega << \omega_{ci}$ limit perpendicular currents associated with AW are provided by the polarization drift of ions $v_{pol,i} = (m_i / eB^2)\partial E_\perp / \partial t$ note that polarization drift due to electrons can be neglected ($m_e << m_i$).

Thus, for the perpendicular AW current we have:

$$\vec{J}_\perp = en\vec{v}_{pol,i} = \frac{nm_i}{B_0^2}\frac{\partial E_\perp}{\partial t} = \frac{1}{\mu_0 v_A^2}\frac{\partial E_\perp}{\partial t} \qquad (4.13)$$

Combining Eqs.(4.13), (4.7) and (4.10) yields:

$$\nabla_\perp \frac{\partial A_z}{\partial z} = -\frac{1}{v_A^2}\nabla_\perp \frac{\partial \varphi}{\partial t} \Rightarrow \frac{\partial A_z}{\partial z} = -\frac{1}{v_A^2}\frac{\partial \varphi}{\partial t} \qquad (4.14)$$

If we apply $\partial / \partial t$ to Eq.(4.12) and use (4.14), we finally obtain a *wave equation* for IAW:

$$\left(1 - \lambda_e^2 \nabla_\perp^2\right)\frac{\partial^2 A_z}{\partial t^2} = v_A^2 \frac{\partial^2 A_z}{\partial z^2} \qquad (4.15)$$

Fourier transform of which gives the dispersion relation for IAW

$$\boxed{\omega^2 = \frac{k_\parallel^2 v_A^2}{\left(1 + \lambda_e^2 k_\perp^2\right)} \qquad (4.16)}$$

Kinetic Alfven Waves (KAWs)

When $\beta >> m_e/m_i$, i.e. $v_A < v_{th,i}, v_{th,e}$ then clearly thermal effects become important. Thus dominant mechanism for sustaining $E_{\|}$ is parallel electron pressure gradient (that is why such waves are called *kinetic* AW – kinetic motion of electrons is source of the pressure).

Thus in Eq.(4.3) for electrons we balance parallel electric field with the pressure gradient term (and ignore the inertia term, we also ignore temperature variation along the magnetic field, because in this direction particle motion is unimpeded, so electrons can thermalise quickly):

$$E_{\|} = -\frac{kT_e}{en_0}\frac{\partial n_e}{\partial z} = -e\mu_0\rho_s^2 v_A^2 \frac{\partial n_e}{\partial z} \qquad (4.17)$$

Here we also used ideal gas law $P_e = nkT_e$ and hence ignored generally tensor nature of electron pressure. $\rho_s = \sqrt{kT_e/m_i}/\omega_{ci}$ is ion Larmor radius at electron temperature. Note that $\rho_s = \rho_i$ when $T_e = T_i$.

From Eqs.(4.6) and (4.17) we obtain:

$$\frac{\partial \varphi}{\partial z} + \frac{\partial A_z}{\partial t} = e\mu_0 \rho_s^2 v_A^2 \frac{\partial n_e}{\partial z} \qquad (4.18)$$

We can use Eq.(4.4) for electrons and (4.9) to obtain auxiliary equation:

$$\frac{\partial n_e}{\partial t} + \frac{1}{e\mu_0} \frac{\partial \nabla_\perp^2 A_z}{\partial z} = 0 \qquad (4.19)$$

which combined with (4.18) and (4.14) gives:

$$\frac{\partial^2 A_z}{\partial t^2} - \frac{\partial}{\partial z} v_A^2 \frac{\partial A_z}{\partial z} + v_A^2 \rho_s^2 \nabla_\perp^2 \frac{\partial^2 A_z}{\partial z^2} = 0 \qquad (4.20)$$

Eq.(4.20) is the *wave equation* for kinetic AWs or KAWs, which upon Fourier transform yields the dispersion relation for KAWs

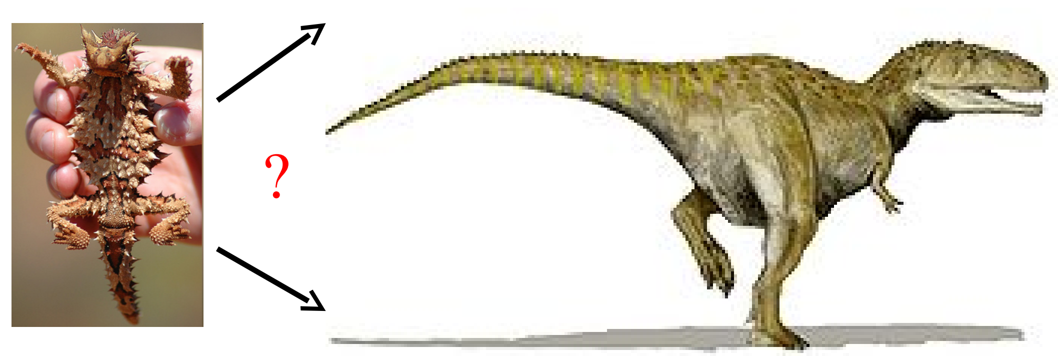
$$\boxed{\omega^2 = k_\parallel^2 v_A^2 \left(1 + \rho_s^2 k_\perp^2\right) \qquad (4.21)}$$

Note that both DAWs, i.e. IAWs and KAWs in the MHD limit, i.e. when $k_\perp \to 0$ recover normal low frequency Alfven waves with usual dispersion relation $\omega^2 = k_\parallel^2 v_A^2$. Cf. **Stasiewicz et al. Sp. Sci. Rev. 92, 423 (2000); Tsiklauri D. Phys. Plasmas 19, 082903 (2012).**

Conclusions:

In the context of coronal heating and particle acceleration (e.g. in solar flares) small scale effects <u>when</u> modelled with correct (full) physics seem to work. Examples: DAWs or X-point collapse (+many more).

The trouble is in the best case we can model 10m^3 of space. Whereas we need to model $10^7 m^3$. Will models work if up-scaled? (re-call why current TOKAMAKs do not work)



We seem to understand each jigsaw puzzle piece, but not a bigger picture, at least as far as <u>modelling</u> goes (even with http://dirac.ac.uk/).

With <u>observations</u> (http://solarprobe.gsfc.nasa.gov/; http://sci.esa.int/solar-orbiter/; https://www.skatelescope.org/) things may have better immediate future prospects.

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SELF-STUDY MATERIAL on EM waves in unmagnetised / magnetised plasmas

Based on "Plasma Dynamics"

by R. O. Dendy (with derivation steps filled in!)

EM waves in magnetised plasmas

Dielectric description of plasma — Unmagnetised

If a medium is subjected to electric field, its macroscopic response is determined by the sum of microscopic responces of individual particles — separation of positive and negative charges.

At macroscopic level, current $\vec{j}$ is

$$\vec{j} = \overleftrightarrow{\sigma}\vec{E} \qquad (1)$$

where $\overleftrightarrow{\sigma}$ is the conductivity tensor.

From Maxwell's Eqs: (This is the displacement current)

$$\frac{1}{\mu_0}\nabla\times\vec{B} = \vec{j} + \varepsilon_0\frac{\partial\vec{E}}{\partial t} \qquad (2)$$

$$\nabla\times\vec{B} = \mu_0\vec{j} + \varepsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} = \mu_0\overleftrightarrow{\sigma}\vec{E} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} \Longrightarrow$$

(Here used Eq. (1) and $\mu_0\varepsilon_0 = 1/c^2$). Then $\vec{E} \sim e^{-i\omega t}$

$$\nabla \times \vec{B} = \left(\mu_0 \overleftrightarrow{\sigma} + \frac{\overleftrightarrow{I}}{c^2}\frac{\partial}{\partial t}\right)\vec{E} = \left(\mu_0 \overleftrightarrow{\sigma} + \frac{\overleftrightarrow{I}}{c^2}(-i\omega)\right)\vec{E} \Rightarrow$$

$$\nabla \times \vec{B} = -\frac{i\omega}{c^2}\left(\overleftrightarrow{I} + \frac{\overleftrightarrow{\sigma}\mu_0 c^2}{(-i\omega)}\right)\vec{E} = -\frac{i\omega}{c^2}\left(\overleftrightarrow{I} + \frac{i\overleftrightarrow{\sigma}}{\varepsilon_0 \omega}\right)\vec{E} \quad (3)$$

$$\boxed{\overleftrightarrow{\varepsilon} = \overleftrightarrow{I} + \frac{i\overleftrightarrow{\sigma}}{\varepsilon_0 \omega}} \quad (4)$$

$\overleftrightarrow{\varepsilon}$ Dielectric tensor

$$\boxed{\nabla \times \vec{B} = -\frac{i\omega}{c^2}\overleftrightarrow{\varepsilon}\vec{E}} \quad (5)$$

Apply $\frac{\partial}{\partial t}$ on Eq. (5), use $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ and

$$\nabla^2 \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla \times \nabla \times \vec{A}$$

$$\nabla \times \frac{\partial \vec{B}}{\partial t} = -\nabla \times \nabla \times \vec{E} = \nabla^2 E - \nabla(\nabla \cdot \vec{E}) =$$

$$= -\frac{i\omega}{c^2} \overleftrightarrow{\varepsilon} \frac{\partial \vec{E}}{\partial t} = -\frac{i\omega}{c^2} \overleftrightarrow{\varepsilon} (-i\omega) \vec{E}$$

$$\nabla^2 \vec{E} - \nabla(\nabla \cdot \vec{E}) + \frac{\omega^2}{c^2} \overleftrightarrow{\varepsilon} \vec{E} = 0 \qquad (6)$$

Consider an electron in $\vec{E}$-field $\vec{B} = 0$ and

$$\vec{E} \sim e^{-i\omega t}$$

$$m\ddot{\vec{x}} = -e\vec{E} \ (\text{Eq. motion}) \quad m\dot{\vec{V}} = -e\vec{E}$$

$$-i\omega m \vec{V} = -e\vec{E}; \qquad \vec{V} = \frac{e}{i\omega m} \vec{E} \qquad (7)$$

Current density is

$$\vec{j} = -n_0 e \vec{v} = -\frac{n_0 e^2}{i\omega m}\vec{E}. \qquad (8)$$

Then according to Eq. (1) $\overleftrightarrow{\sigma} = -\frac{n_0 e^2}{i\omega m}\overleftrightarrow{I}$

Substituting into Eq. (4)

$$\overleftrightarrow{\varepsilon} = \overleftrightarrow{I} + \frac{i}{\varepsilon_0 \omega}\left(-\frac{n_0 e^2}{i\omega m}\right)\overleftrightarrow{I} = \left(1 - \frac{\omega_{pe}^2}{\omega^2}\right)\overleftrightarrow{I} \qquad (9)$$

where $\omega_{pe} = \sqrt{\frac{n_0 e^2}{\varepsilon_0 m}}$ is the electron plasma freq.

Normal modes of cold, unmagnetised plasma can be calculaded as follows:

sub. Eq (9) into Eq. (6)

$$\nabla^2 \vec{E} - \nabla(\nabla \cdot \vec{E}) + \frac{\omega^2}{c^2}\left(1 - \frac{\omega_{pe}^2}{\omega^2}\right)\overleftrightarrow{I}\vec{E} = 0$$

and assume $\vec{E} \sim e^{i(\vec{k}\vec{r} - \omega t)}$

$$-k^2c^2\vec{E}+\vec{k}(\vec{k}\cdot\vec{E})c^2+\omega^2\vec{E}-\omega_{pe}^2\vec{E}=0$$

$$(\omega^2-\omega_{pe}^2-k^2c^2)\vec{E}+c^2\vec{k}(\vec{k}\cdot\vec{E})=0 \qquad (10)$$

There are two types of waves (normal modes):

1) transverse electromagnetic, that have zero projection on wavenumber $\vec{k}$, i.e. $\vec{E}\cdot\vec{k}=0$

for these waves based on Eq. (10):

$$(\omega^2-\omega_{pe}^2-k^2c^2)\vec{E}=0 \qquad (11)$$

For non-zero $\vec{E}$ this true is

$$\omega^2=\omega_{pe}^2+k^2c^2 \qquad (12)$$

This indicates that in non-magnetized plasmas EM wave frequency should alway exceed ω_{pe}

Otherwise wave is evanescent (non-propagating).

2) longitudinal electrostatic, which can be obtained by projecting Eq. (10) on $\vec{K}$ i.e. multiply Eq. (10) <u>scalarly</u> by $\vec{K}$

$$(\omega^2-\omega_{pe}^2-K^2c^2)\cancel{(\vec{K}\cdot\vec{E})}+c^2(\vec{K}\cdot\vec{K})\cancel{(\vec{K}\cdot\vec{E})}=0$$

$$\omega^2-\omega_{pe}^2-K^2c^2+K^2c^2=0$$

$$\omega^2=\omega_{pe}^2 \qquad (13)$$

Dielectric description of Plasma – Magnetised

Now consider an electron in $\vec{E}$-field again but $\vec{B} \neq 0$.

Then eq. of motion has two components,

parallel to the $\vec{B}$ – field

$$\dot{\vec{V}}_{\parallel} = -\frac{e}{m}\vec{E}_{\parallel}(t) \qquad (14)$$

and perpendicular to it:

$$\dot{\vec{V}}_{\perp} = -\frac{e}{m}\vec{E}_{\perp}(t) + \vec{\omega}_{ce} \times \vec{V}_{\perp} \qquad (15)$$

Differentiate Eq. (15) w.r.t. time and use Eq. (14):

$$\ddot{\vec{V}}_{\perp} = -\frac{e}{m}\dot{\vec{E}}_{\perp} + \vec{\omega}_{ce} \times \dot{\vec{V}}_{\perp} = -\frac{e}{m}\dot{\vec{E}}_{\perp} - \frac{e}{m}\vec{\omega}_{ce} \times \vec{E}_{\perp} + \overset{A}{\vec{\omega}_{ce}} \times (\overset{B}{\vec{\omega}_{ce}} \times \overset{C}{\vec{V}_{\perp}}) =$$

$$= -\frac{e}{m}\left(\dot{\vec{E}}_{\perp} + \vec{\omega}_{ce} \times \vec{E}_{\perp}\right) + \vec{\omega}_{ce}\,\cancelto{0}{(\vec{\omega}_{ce} \cdot \vec{V}_{\perp})} - \vec{V}_{\perp}(\vec{\omega}_{ce} \cdot \vec{\omega}_{ce})$$

$$\ddot{\vec{V}}_\perp + \omega_{ce}^2 \vec{V}_\perp = -\frac{e}{m}\left(\dot{\vec{E}}_\perp + \vec{\omega}_{ce} \times \vec{E}_\perp\right) \qquad (16)$$

where in the 2nd line we used $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$

Eq. (16) shows that in the absence of the $E_\perp$ (R.H.S. = 0) it reduces to the previous helix solution perpendicular component which we denote by $\vec{V}_{\perp 0}(t)$

Now assume $\vec{E}_\perp(t) = Re(\tilde{\vec{E}}_\perp e^{-i\omega t})$ and

$$\vec{V}_\perp(t) = \vec{V}_{\perp 0}(t) + Re(\tilde{\vec{V}}_\perp e^{-i\omega t})$$

Then Eq. (16) becomes:

$$(\omega_{ce}^2-\omega^2)\tilde{V}_\perp = -\frac{e}{m}\left(-i\omega\tilde{E}_\perp + \vec{\omega}_{ce}\times\tilde{E}_\perp\right) \qquad (17)$$

We can now combine $V_\perp$ and $V_\parallel$ into a single vector

$$\vec{V}(t) = \vec{V}_{\perp 0}(t) + \mathrm{Re}\left(\overleftrightarrow{a}\,\tilde{E}\,e^{-i\omega t}\right) \qquad (18)$$

where $\overleftrightarrow{a}$ is

$$\overleftrightarrow{a} = \frac{e}{m}\begin{pmatrix} \frac{i\omega}{\omega_{ce}^2-\omega^2} & \frac{\omega_{ce}}{\omega_{ce}^2-\omega^2} & 0 \\ \frac{-\omega_{ce}}{\omega_{ce}^2-\omega^2} & \frac{i\omega}{\omega_{ce}^2-\omega^2} & 0 \\ 0 & 0 & i/\omega \end{pmatrix} \qquad (19)$$

This follows from the fact that (from Eq. (17))

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{K} \\ 0 & 0 & \omega_{ce} \\ E_x & E_y & E_z \end{vmatrix}$$

$$\left.\begin{aligned}
\tilde{\vec{V}}_\perp &= \frac{e}{m(\omega_{ce}^2-\omega^2)}\left(i\omega\tilde{E}_\perp - \vec{\omega}_{ce}\times\tilde{E}_\perp\right)\\
V_x &= \frac{e}{m(\omega_{ce}^2-\omega^2)}\left(i\omega\tilde{E}_x + \omega_{ce}\tilde{E}_y\right)\\
V_y &= \frac{e}{m(\omega_{ce}^2-\omega^2)}\left(i\omega\tilde{E}_y - \omega_{ce}\tilde{E}_x\right)\\
V_z &= V_\parallel = -\frac{i}{\omega}\frac{e}{m}\tilde{E}_z
\end{aligned}\right\} \quad (20)$$

In the cold plasma approximation there are no random thermal motions, therefore in Eq. (18) $\vec{V}_{\perp 0}(t) = 0$.

Then the current associated with the response of electrons to the EM wave is

$$\vec{j}(t) = -n_0 e \vec{v}(t) = -n_e e \overleftrightarrow{a}\vec{E} = \overleftrightarrow{\sigma}\vec{E}(t) \quad (21)$$

Recalling that $\overleftrightarrow{\varepsilon} = \overleftrightarrow{I} + \frac{i\overleftrightarrow{\sigma}}{\varepsilon_0\omega}$, we have for the dielectric tensor $\overleftrightarrow{\varepsilon}$:

$$\overleftrightarrow{\varepsilon} = \begin{pmatrix} 1 + \frac{e}{m}(-n_0 e)\frac{i^2 \cancel{\omega}}{\varepsilon_0 \cancel{\omega}}\frac{1}{\omega_{ce}^2-\omega^2}, & \frac{e}{m}(-n_0 e)\frac{i\omega_{ce}}{\varepsilon_0\omega}\frac{1}{\omega_{ce}^2-\omega^2}, & 0 \\ \frac{e}{m}(-n_0 e)\frac{i}{\varepsilon_0\omega}\frac{(-\omega_{ce})}{\omega_{ce}^2-\omega^2}, & 1 + \frac{e}{m}(-n_0 e)\frac{i}{\varepsilon_0 \cancel{\omega}}\frac{i\cancel{\omega}}{(\omega_{ce}^2-\omega^2)}, & 0 \\ 0, & 0, & 1 + \frac{e}{m}\left(-\frac{i}{\omega}\right)\frac{i}{\varepsilon_0\omega}(-n_0 e) \end{pmatrix}$$

$$\omega_{pe} \equiv \sqrt{\frac{n_0 e^2}{\varepsilon_0 m}}$$ — electron plasma frequency

$$\overleftrightarrow{\varepsilon} = \begin{pmatrix} 1 + \frac{\omega_{pe}^2}{\omega_{ce}^2-\omega^2}, & -\frac{i\omega_{ce}}{\omega}\frac{\omega_{pe}^2}{\omega_{ce}^2-\omega^2}, & 0 \\ \frac{i\omega_{ce}}{\omega}\frac{\omega_{pe}^2}{\omega_{ce}^2-\omega^2}, & 1 + \frac{\omega_{pe}^2}{\omega_{ce}^2-\omega^2}, & 0 \\ 0, & 0, & 1 - \frac{\omega_{pe}^2}{\omega^2} \end{pmatrix} \quad (22)$$

In order to determine what are the normal modes in the cold, magnetised plasma, we need to use Eq.(6), i.e

$$\nabla^2 \vec{E} - \nabla(\nabla \cdot \vec{E}) + \frac{\omega^2}{c^2} \overleftrightarrow{\varepsilon} \vec{E} = 0$$

plug Eq.(22) into it, and sub. harmonic solutions of the type $e^{i\vec{k}\vec{r} - i\omega t}$:

$$\left(-k^2 \overleftrightarrow{I} + \vec{k}\vec{k} + \frac{\omega^2}{c^2} \overleftrightarrow{\varepsilon}\right) \vec{E} = 0 \text{ or} \qquad (23)$$

in component notation (repeated index summation assumed)

$$\left(-k^2 \delta_{ij} + k_i k_j + \frac{\omega^2}{c^2} \varepsilon_{ij}\right) E_j = 0 \qquad (24)$$

Let us first simplify notation by re-writing Eq.(22)

$$\overleftrightarrow{\varepsilon} = \varepsilon_{ij} = \begin{pmatrix} \varepsilon_{\perp}, & -ig, & 0 \\ ig, & \varepsilon_{\perp}, & 0 \\ 0, & 0, & \varepsilon_{\parallel} \end{pmatrix}, \quad \text{~~where~~} \qquad (24^*)$$

where

$$\varepsilon_{\perp} = 1 + \frac{\omega_{pe}^2}{\omega_{ce}^2 - \omega^2}; \quad g = \frac{\omega_{ce}}{\omega}\frac{\omega_{pe}^2}{\omega_{ce}^2 - \omega^2}; \quad \varepsilon_{\parallel} = 1 - \frac{\omega_{pe}^2}{\omega^2}$$

Eq.(23) or (24) can be written in matrix form

$$\overleftrightarrow{\Lambda}\vec{E} = 0 \qquad (25)$$

where $\overleftrightarrow{\Lambda} = \frac{\omega^2}{c^2}\varepsilon_{ij} + K_i K_j - K^2 \delta_{ij} = \frac{\omega^2}{c^2}\overleftrightarrow{\varepsilon} + \vec{K}\vec{K} - K^2 \overleftrightarrow{I}$

As we will see below Eq.(25) defines <u>polarisation</u> of the normal mode, i.e. how (in which direction) electric field vector oscillates.

The <u>normal modes</u> of the system are determined from the mathematical compatibility of the system of equations (25) — there are 3 equations, x, y, z components, of course. The compatibility condition is

$$\det(\overleftrightarrow{\Lambda}) = \| \overleftrightarrow{\Lambda} \| = 0 \qquad (26)$$

i.e. determinant of the matrix $\overleftrightarrow{\Lambda}$ being zero.

Eq. (26) is a general compatibility condition of algebraic equations — this is <u>true</u> for <u>any</u> system of equations.

If we substitute Eq. (24*) into Eq. (26), also introducing the notation $\vec{N} = c\vec{k}/\omega$, we obtain

$$\det \frac{\omega^2}{c^2} \begin{pmatrix} \varepsilon_1 - N_y^2 - N_z^2, & -ig + N_x N_y, & N_x N_z \\ ig + N_x N_y, & \varepsilon_1 - N_x^2 - N_z^2, & N_y N_z \\ N_x N_z, & N_y N_z, & \varepsilon_{11} - N_x^2 - N_y^2 \end{pmatrix} = 0 \quad (27)$$

Eq. (27) shows that all three normal modes are generally inter-coupled. However in the simplest case of propagation along the magnetic field it is possible to have longitudinal modes decoupled from the transverse ones.

In the case of parallel to the magnetic field propagation $N_x = N_y = 0$ $N_z = N$ $(\vec{B} \parallel Oz)$

Then Eq. (27) reduces to

$$\det \frac{\omega^2}{c^2} \begin{pmatrix} \varepsilon_\perp - N^2, & -ig, & 0 \\ ig, & \varepsilon_\perp - N^2, & 0 \\ 0, & 0, & \varepsilon_\parallel \end{pmatrix} =$$

$$= (\varepsilon_\perp - N^2)(\varepsilon_\perp - N^2)\varepsilon_\parallel + ig \cdot ig \cdot \varepsilon_\parallel =$$

$$= \varepsilon_\parallel \left((\varepsilon_\perp - N^2)^2 - g^2 \right) = 0 \qquad (28)$$

Eq (28) clearly shows ~~it~~ can be satisfied if

$\varepsilon_\parallel = 0$ (longitudinal mode)

$(\varepsilon_\perp - N^2)^2 - g^2 = 0$ (two transverse modes)

From Eq. (24*) $\varepsilon_{\parallel}=0$ normal mode implies

$$1-\frac{\omega_{pe}^2}{\omega^2}=0 \Rightarrow \omega^2=\omega_{pe}^2 \tag{29}$$

because $(\varepsilon_{\perp}-N^2)^2-g^2\neq 0$,

$$\begin{pmatrix} \varepsilon_{\perp}-N^2, & -ig, & 0 \\ ig, & \varepsilon_{\perp}-N^2, & 0 \\ 0, & 0, & \varepsilon_{\parallel} \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}=0 \Rightarrow$$

$$\begin{pmatrix} (\varepsilon_{\perp}-N^2), & -ig \\ ig, & \varepsilon_{\perp}-N^2 \end{pmatrix}\begin{pmatrix} E_x \\ E_y \end{pmatrix}+\varepsilon_{\parallel}E_z=0 \tag{30}$$

can be satisfied only by $\underline{(0,0,E_z)}$.

This proves $\varepsilon_{\parallel}=0$ is the longitudinal, electrostatic mode. i.e presence of the magnetic field does not alter the mode properties in the parallel to the B-field direction — recall Eq.(13) — it is identical to Eq.(29).

Now consider $\varepsilon_{\parallel}\neq 0$, $(\varepsilon_{\perp}-N^2)^2-g^2=0$ case

This has two solutions (two normal modes)

$$\varepsilon_{\perp}-N^2=\pm g.$$

First consider "upper" sign ("+") (look up Eq.(24*))

$$\varepsilon_{\perp}-g=N^2 \Rightarrow$$

$$1+\frac{\omega_{pe}^2}{\omega_{ce}^2-\omega^2}-\frac{\omega_{ce}}{\omega}\frac{\omega_{pe}^2}{\omega_{ce}^2-\omega^2}=\frac{c^2K^2}{\omega^2}$$

$$1-\frac{\omega_{pe}^2(\omega_{ce}-\omega)}{\omega(\omega_{ce}^2-\omega^2)}=1-\frac{\omega_{pe}^2\cancel{(\omega_{ce}-\omega)}}{\omega(\omega_{ce}+\omega)\cancel{(\omega_{ce}-\omega)}}=\frac{c^2K^2}{\omega^2}\Rightarrow$$

$$K = \frac{\omega}{c}\sqrt{1 - \frac{\omega_{pe}^2}{\omega(\omega_{ce}+\omega)}} \quad (31)$$

Eq. (31) gives us the dispersion relation, i.e. relation between ω and K.

Now, let us examine the polarisation of this mode for this we use Eq (25) or better Eq. (30).

Eq. (30) can be satisfied only by $(E_x, E_y, 0)$ with $\varepsilon_\perp - N^2 = g$. i.e. $N^2 = \varepsilon_\perp - g \rightarrow$ Eq. (30)

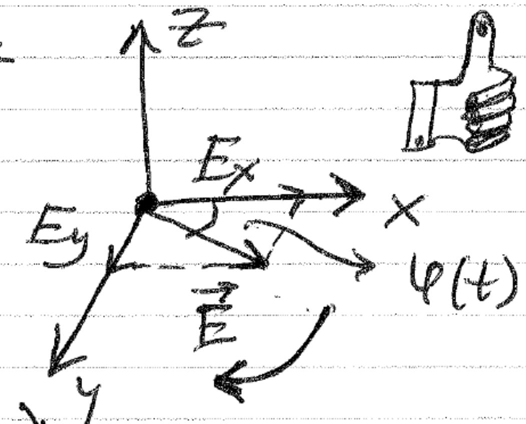
$$E_x(\cancel{\varepsilon_\perp} - \cancel{\varepsilon_\perp} + \cancel{g}) + E_y(-i\cancel{g}) = 0 \Rightarrow$$

$$E_x - iE_y = 0 \;; \quad E_x = iE_y \quad (32)$$

What does Eq. (32) mean geometrically?

$$E_x = Re\left(\tilde{E}\, e^{i(k_z z - \omega t)}\right) =$$

$$= Re\left(\tilde{E}\, e^{i\varphi(t)}\right) \sim \cos\varphi(t)$$



$$E_y = -i Re\left(\tilde{E}\, e^{i\varphi(t)}\right) =$$

$$= Re\left((-i)\tilde{E}\cos\varphi(t) + (-i)\tilde{E} i \sin\varphi(t)\right) =$$

$$= Re\left(-i\tilde{E}\cos\varphi(t) + \tilde{E}\sin\varphi(t)\right) \sim \sin\varphi(t) =$$

$$= \cos\left(\frac{\pi}{2} - \varphi(t)\right)$$

at $t=0$ $\vec{E} \parallel OX$ then

at $t>0$ E_y grows in positive y direction

i.e. this solution gives left-hand circularly polarised wave. If left hand thumb is along z-axis, curled fingers show direction of rotation of the E-field vector.

Then the "lower", "−" sign gives

$$\varepsilon_{\perp} - N^2 = -g \text{ , i.e } \varepsilon_{\perp} + g = N^2$$

Similar to the above analysis in this case yields: the following dispersion relation

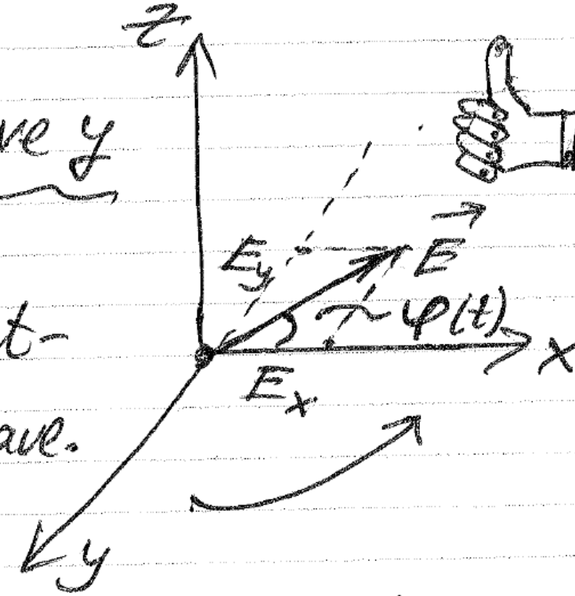
$$k = \frac{\omega}{c}\sqrt{1 + \frac{\omega_{pe}^2}{\omega(\omega_{ce} - \omega)}} \qquad (33)$$

Eq. (30) gives the polarisation of this mode $(E_x, E_y, 0)$ with $\varepsilon_{\perp} - N^2 = -g \Rightarrow E_x = -iE_y$ (34)

i.e. $E_y = iE_x \Rightarrow$

$$E_y = Re\left(i\tilde{E}\cos\varphi(t) - \tilde{E}\sin\varphi(t)\right) \sim -\sin\varphi(t)$$

at $t=0$ $\vec{E} \parallel Ox$ then
at $t>0$ E_y grows in negative y direction.



Thus now solution gives right-hand circularly polarased wave.

If right hand thumb is along z-axis, curled fingers show the direction of rotation of the $\vec{E}$-field vector.

Note that Eq. (33) becomes singular as $\omega \to \omega_{ce}$. This happens because in the right-hand circularly polarised wave, the electric field vector rotates in the same direction as the electron on its helical trajectory. Therefore when $\omega \to \omega_{ce}$ electrons come in resonance with the waves.

This electron cyclotron resonance leads to a rapid damping of the wave.

Before presenting overall picture of the available normal mode dispersion, let us make our calculations more realistic by including (allowing) the motion of ions which were all ignored.

Our Eq. (24*) contained quantities such as ω_{pe} and ω_{ce}. Naturally if we include dynamics of ions, we should expect that the following frequencies to be present too:

$$\omega_{pi} = \sqrt{\frac{n_i (Ze)^2}{\varepsilon_0 m_i}} \quad \left(\text{in analogy with } \omega_{pe} = \sqrt{\frac{n_e e^2}{\varepsilon_0 m_e}}\right)$$

Note that plasma quasineutrality condition also implies

$$n_e = Z n_i \Rightarrow n_i = n_e / Z \Rightarrow$$

$$\omega_{pi} = \sqrt{\frac{n_e Z e^2}{\varepsilon_0 m_i}}; \qquad (35)$$

and $\omega_{ci} = -\frac{ZeB}{m_i}$ (36) (in analogy with $\omega_{ce} = \frac{eB}{m_e}$)

Note the "$-$" sign because of the opposite sign of ions.

ω_{pi} is called ion plasma frequency and ω_{ci} is ion cyclotron frequency.

It can be shown that if we were to include dynamics of ions in Eqs. (14) and (15) we would simply end up in lieu of Eq. (24*) with the following:

$$\varepsilon_\perp = 1 + \frac{\omega_{pe}^2}{\omega_{ce}^2 - \omega^2} + \frac{\omega_{pi}^2}{\omega_{ci}^2 - \omega^2} = 1 + \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega_{c\alpha}^2 - \omega^2}$$

$$g = \frac{\omega_{ce}}{\omega}\frac{\omega_{pe}^2}{\omega_{ce}^2 - \omega^2} - \frac{\omega_{ci}}{\omega}\frac{\omega_{pi}^2}{\omega_{ci}^2 - \omega^2} = \sum_\alpha \frac{\omega_{c\alpha}}{\omega}\frac{\omega_{p\alpha}^2}{\omega_{c\alpha}^2 - \omega^2}$$

$$\varepsilon_\parallel = 1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{pi}^2}{\omega^2} = 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2}$$

The latter equations are for the general case of α - species of particles.

Thus for the longitudinal, electrostatic mode we now have

$$\varepsilon_{\parallel}=0 \Rightarrow 1-\frac{\omega_{pe}^2}{\omega^2}-\frac{\omega_{pi}^2}{\omega^2}=1-\frac{\omega_{pe}^2}{\omega^2}\left(1+\frac{\omega_{pi}^2}{\omega_{pe}^2}\right)=0$$

Because $\frac{\omega_{pi}^2}{\omega_{pe}^2}=\frac{Zm_e}{m_i}\sim\frac{1}{1836}\ll 1$ for electron-proton (hydrogen) plasma

we see that inclusion of ion dynamics has only a minor role on the longitudinal mode by slightly increasing the plasma cut-off frequency.

The situation is different in the case of transverse modes (in the case of parallel to $\vec{B}$ propagation. Let us consider left-hand polarised wave.

Now, instead of Eq.(31) we have

$$K = \frac{\omega}{c}\sqrt{1 - \frac{\omega_{pe}^2}{\omega(\omega_{ce}+\omega)} + \frac{\omega_{pi}^2}{\omega_{ci}^2-\omega} + \frac{\omega_{ci}}{\omega}\frac{\omega_{pi}^2}{\omega_{ci}^2-\omega^2}} =$$

$$= \frac{\omega}{c}\sqrt{1 - \frac{\omega_{pe}^2}{\omega(\omega_{ce}+\omega)} + \frac{\omega_{pi}^2\cancel{(\omega_{ci}+\omega)}}{\omega(\omega_{ci}-\omega)\cancel{(\omega_{ci}+\omega)}}} =$$

$$= \frac{\omega}{c}\sqrt{1 - \frac{\omega_{pe}^2}{\omega(\omega_{ce}+\omega)} + \frac{\omega_{pi}^2}{\omega(\omega_{ci}-\omega)}} =$$

$$= \frac{\omega}{c}\sqrt{1 + \frac{\omega_{pi}^2(\omega_{ce}+\omega) - \omega_{pe}^2(\omega_{ci}-\omega)}{\omega(\omega_{ci}-\omega)(\omega_{ce}+\omega)}} \qquad (37)$$

Now note that $\frac{\omega_{pe}^2}{\omega_{ce}} = \frac{\omega_{pi}^2}{\omega_{ci}}$. This is easy to check

$$\frac{n_e e^2}{\varepsilon_0 \cancel{m_e}}\frac{\cancel{m_e}}{eB} = \frac{n_e \cancel{Z} e^2}{\varepsilon_0 \cancel{m_i}}\frac{\cancel{m_i}}{\cancel{Z} eB} \Longrightarrow$$

$$\omega_{pe}^2\omega_{ci} = \omega_{pi}^2\omega_{ce} \qquad (38)$$

Thus Eq. (37) can be continued as

$$K = \frac{\omega}{c}\sqrt{1 + \frac{\underline{\omega_{pi}^2 \omega_{ce}} + \omega_{pi}^2 \cancel{\omega} - \underline{\omega_{pe}^2 \omega_{ci}} + \omega_{pe}^2 \cancel{\omega}}{\cancel{\omega}(\omega_{ci} - \omega)(\omega_{ce} + \omega)}}$$

$$K = \frac{\omega}{c}\sqrt{1 + \frac{\omega_{pe}^2 + \omega_{pi}^2}{(\omega_{ce} + \omega)(\omega_{ci} - \omega)}} \qquad (39)$$

Note that Eq. (39) becomes singular when $\omega \to \omega_{ci}$.
<u>ALSO</u> note that Eq. (39) is generalisation of Eq (31) by including <u>ion dynamics</u>. Thus we conclude that inclusion of ion dynamics induces appearance of <u>ion-cyclotron resonance</u>.
Recall that Eq. (39) is for the <u>left</u> circular polarisation. Thus in this wave electric field vector rotates in the same direction as <u>ions</u> do.
Thus wave comes in resonance with <u>ions</u>.

Let us consider low frequency limit of Eq. (39)

i.e. $\omega \ll \omega_{ci}$ which also guarantees that $\omega \ll \omega_{ce}$

also note that $\omega_{pi} \ll \omega_{pe}$

Thus Eq. (39) reduces to

$$K = \frac{\omega}{c}\sqrt{1 + \frac{\omega_{pe}^2}{\omega_{ce}\omega_{ci}}},$$

but recall that $\omega_{pe}^2 / \omega_{ce} = \omega_{pi}^2 / \omega_{ci} \Longrightarrow$

$$K = \frac{\omega}{c}\sqrt{1 + \frac{\omega_{pi}^2}{\omega_{ci}^2}}$$

but $$\frac{\omega_{pi}}{\omega_{ci}} = \sqrt{\frac{n_e Z e^2}{\varepsilon_0 m_i}} \cdot \frac{m_i}{ZeB} = \frac{\sqrt{\mu_0 n_i m_i}\, c}{B} \quad (40)$$

where we used $n_e = Z n_i$; $\frac{1}{\sqrt{\varepsilon_0}} = c\sqrt{\mu_0}$.

But $\frac{B}{\sqrt{\mu_0 n_i m_i}}$ is the Alfven speed, V_A.

Thus we have

$$K = \frac{\omega}{c}\sqrt{1 + \frac{\omega_{pi}^2}{c\omega_{ci}^2}} = \frac{\omega}{c}\sqrt{1 + \frac{c^2}{V_A}} \Rightarrow$$

$$\frac{\omega}{K} = \frac{c}{\sqrt{1 + \frac{c^2}{V_A^2}}} = \frac{c}{\frac{c}{V_A}\sqrt{1 + \frac{V_A^2}{c^2}}} = \frac{V_A}{\sqrt{1 + V_A^2/c^2}} \quad (41)$$

It is easy to show that $\frac{1}{\sqrt{1 + V_A^2/c^2}}$ appears in our equations because of the displacement current in the Maxwell equations.

In most space/solar plasma problems $V_A/c \ll 1$

so $\frac{\omega}{k} = V_{phase} \approx V_A$ implies that left circularly polarised wave in the low frequency limit lands on the Alfven wave branch and cuts off on ω_{ci}.

We can do similar analysis for the right circularly polarised wave as well, which yields

$$k = \frac{\omega}{c}\sqrt{1 + \frac{\omega_{pe}^2 + \omega_{pi}^2}{(\omega_{ce} - \omega)(\omega_{ci} + \omega)}} \qquad (42)$$

Home work: (1) derive Eq. (42)

(2) derive low frequency $\omega \ll \omega_{ci}$ limit

In class question: what cut off Eq. (42) has and why?