# Harmonic Probability Weighting Functions And Resolution of The Preference Reversal Puzzle

**G. Charles-Cadogan**
**University of Leicester**

**Presentation:**
**Models to Decisions (M2D)**
**1st Annual Conference on Decision Making Under Uncertainty**
**University of Exeter**
**11 July 2017 - 14 July 2017**

# Summary

- What does the paper do? It introduces **a weak harmonic transitivity (WHT) axiom** that supports an abstract harmonic probability weighting function (HPWF). The *abstract HPWF is a random field that mimics the mental states of decision makers (DMs)*. A coherent functional specification is derived via entropy analysis.

- How does the paper contribute to the literature?
  - It introduces a novel concept of **probability cycles in decision making** based on **a phase function of $Z$-score (standardized score) of payoffs**.
  - It shows how observer interference with DMs anchor probability distribution for a set of outcomes can cause them to report preference reversal (PR) when there is none.
  - It resolves the **probability interference factor**, popularized in quantum probability theory, using a classic Kolmogorov probability framework.
  - Our results easily extend to Hilbert space which is the domain of quantum probability theory

- How does the function perform empirically? It *resolves the PR puzzle*, conjunction fallacy, etc. Harmonic regressions can be used to estimate it. Heteroscedasticity correction factor for harmonic regression recovers linear probabilities so it is a **debiasing factor**.

# Outline

- Introduction
- Cognitive basis for harmonics in decision making–illustrations
- Foundations of abstract harmonic probability weighting function (HPWF)
  - Weak harmonic transitivity axiom
  - Topology of attaching mental states to outcomes
- Applications
  - Simulated HPWF with simple phase functions
  - Coherent HPWF from maximum entropy methods
  - Harmonic regression specification of HPWF
  - Behavioural matrix operator for pwf shape
  - Behavioural dynamics for pwf phase shift
- HPWF Resolution of preference reversal
- Conclusion

# Introduction

# HPWF theory and its motivation

- An important issue in decision theory is how to model unobserved noise and imprecision in decision making in the presence of risk.
- Every stochastic process or random variable has a harmonic representation [Bochner, 1955]. However, decision theory neglects this dual harmonic representation of noise.
- WHT axiom is a dual to the weak stochastic transitivity axiom popularized by [Debreu, 1958, Tversky, 1969].
- Harmonic brain wave patterns reflect cognitive states so they should be reflected in pwfs.
- Probability cycles in HPWF mimic human phase response curve (PRC). See [Canavier, 2006] for review of PRC. Can be estimated via harmonic regression. They also mimic the principle of bounded subadditivity that turns: (**phase 1**) impossibility into possibility, (**phase 2**) possibility into likelihood, and (**phase3**) likelihood into certainty in the state space for probability weighting functions [Tversky and Wakker, 1995, Tversky and Fox, 1995].

# Preference reversal puzzle

# Classic preference reversal

The following example is taken from [Goldstein and Einhorn, 1987, pp. 236-237].

- Experiment E1 Subjects choose between bets: one of which has a high probability of winning a small amount of money ($P$-bet), and the other has a low probability of winning a large amount of money (\$-Bet):

$$P\text{-bet: Win \$4 with } p = 0.97, \text{ Lose \$l with } p = 0.03$$
$$\text{\$-bet: Win \$16 with } p = 0.31, \text{ Lose \$1.5 with } p = 0.69$$

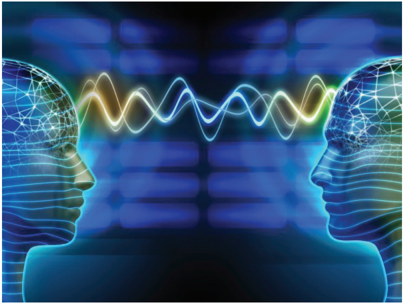
  When subjects are asked to choose which gamble they prefer to play, approximately half choose the $P$-bet over the \$-bet.

- Experiment E2 Each gamble is now presented singly and subjects are asked to state the lowest selling price for the gamble if they owned it, the \$-bet receives a higher price than the $P$-bet. If selling prices reflect preferences, then preference reverses depending on whether subjects chooses or states selling prices.

# Harmonics of Mental and Cognitive states–illustrations

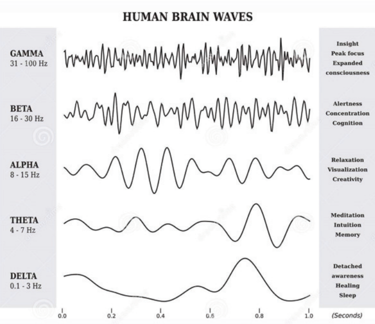
# Neuronal and cognitive states for brain wave harmonics

Figure: Neural Brain Waves



Source:
http://betanews.com/2014/06/11/

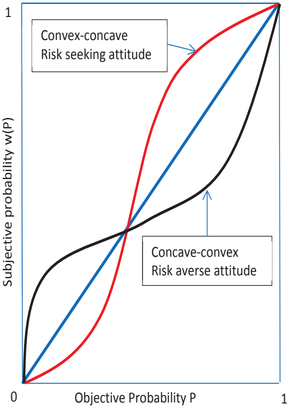
Figure: Human Brain Waves




Source:
https://www.dreamstime.com

Human brain waves are characterized by harmonic frequencies in mental/cognitive states

# Probabilistic risk attitudes



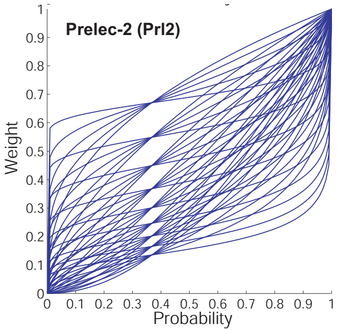

**Primitives**:
$\boldsymbol{x} = (x_1, \ldots, x_n)$ is a distribution of outcomes or payoffs;
$P(\boldsymbol{x}) = (p_1, \ldots, p_n)$ a probability distribution over $\boldsymbol{x}$.
$Q(\boldsymbol{x}) = (q_1, \ldots, q_n)$ a different probability distribution.
$L_1 = (\boldsymbol{x}, P(\boldsymbol{x})) = \big((x_1, p_1), \ldots, (x_n, p_n)\big)$ and $L_2 = (\boldsymbol{x}, Q(\boldsymbol{x}))$
$= \big((x_1, q_1), \ldots, (x_n, q_n)\big)$ are two lotteries
or gambles. Outcomes are ranked ordered from min to max.
Given utility function $U(x)$, $V^{\text{lin}}(P) = \sum_{j=1}^{n} U(x_j)p_j$ and
$V^{\text{lin}}(Q) = \sum_{j=1}^{n} U(x_j)q_j$ are linear functionals in $P$ and $Q$, resp.

- **Expected Utility Theory** (EUT):
  Subjects prefer $P$ to $Q$ iff $V^{\text{lin}}(P) > V^{\text{lin}}(Q)$.
- **Non-Expected Utility Theory** (non-EUT):
  $V^{\text{nonlin}}(P) = \sum_{j=1}^{n} U(x_j)w(p_j)$
  and $V^{\text{nonlin}}(Q) = \sum_{j=1}^{n} U(x_j)w(q_j)$ are nonlinear functionals in $P$ and $Q$, resp. Not clear how subjects choose between $P$ and $Q$ unless $w(P) > w(Q)$ uniformly.

- **Under EUT** probabilistic risk attitudes are "neutral"–reflected along diagonal $w(P) = P$, and risk preference burden falls on utility function $U(x)$.
- **Under non-EUT** probabilistic risk attitudes are "non-neutral"–reflected off diagonal $w(P) \neq P$, and risk preference burden is carried jointly by subjective probability distribution (i.e., probability weighting function) $w(P)$ **and** utility function $U(x)$.
- Study of probability weighting functions is independently important element of evaluating decision making under risk and uncertainty.

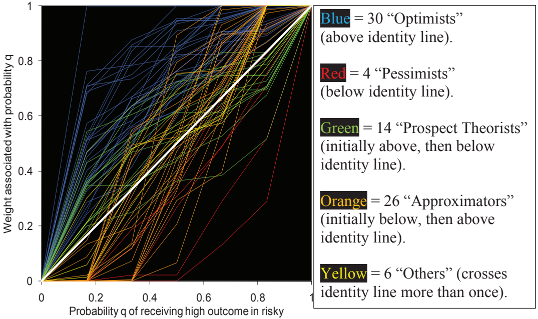
# Probability weighting harmonics

Figure: Prelec 2-factor pwf




Source: [Cavagnaro et al., 2013]

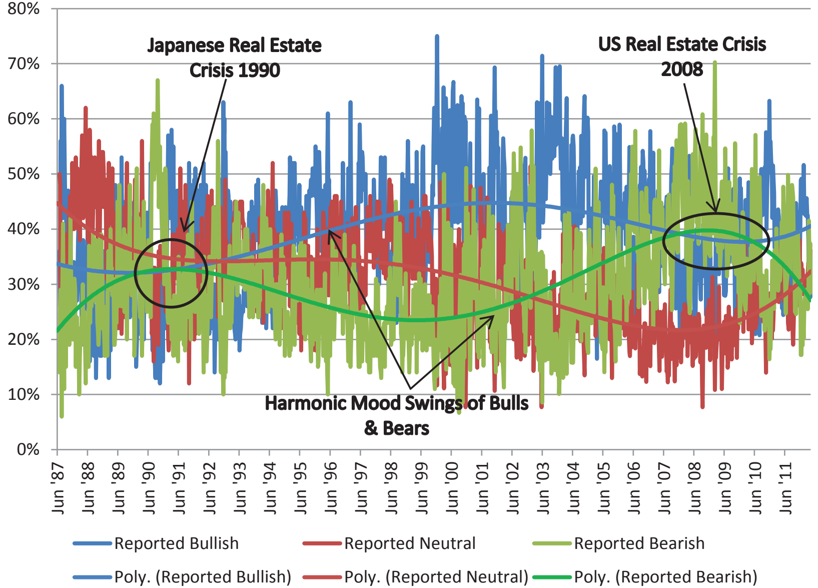
Figure: Wilcox experiments




Source: [Wilcox, 2011]

[Prelec, 1998] posited a pwf $w(p) = \exp\left(-\beta(-\ln(p))^{\alpha}\right)$ where $\alpha > 0, \beta > 0$ are **curvature** and **elevation** parameters, respectively. Axiomatized fixed point probability is $e^{-1}$. **Empirical regularity**: People over weight small probabilities and underweight large probabilities.

# Technical analysis: Mood swings in financial markets




**Source**: Author's analysis of American Association of Individual Investors Weekly Survey data. Markets crash when bulls and bears agree. Cf. [Lo et al., 2000].

# Foundations of Abstract HPWF

# Axiomatic foundations of HPWF

Keys: "strictly preferred to" $(\succ)$; "indifferent from" $(\sim)$; "weakly preferred to" $(\succeq)$

**Canonical expected utility theory (EUT) framework** [Von Neumann and Morgenstern, 1953]

## Axiom (*Completeness*)

*For given $A, B \in X$, wither $A \succeq B$ or $B \succeq A$ or both, i.e., $A \sim B$.*

## Axiom (*Transitivity (ST)*)

*For given $A, B, C \in X$, if $A \succ B$ and $B \succ C$, then $A \succ C$.*

## Axiom (*Independence*)

*For given $A, B, C \in X$ and $\alpha \in (0, 1]$, if $A \succeq B$, then $\alpha A + (1-\alpha)C \succeq \alpha B + (1-\alpha)C$*

**We relax the transitivity axiom**

## Axiom (*Weak Stochastic Transitivity (WST) Axiom*)

*If $P\{A \succ B\} \geq \frac{1}{2}$ and $P\{B \succ C\} \geq \frac{1}{2}$ , then $P\{A \succ C\} \geq \frac{1}{2}$, e.g., [Tversky, 1969].*

## Axiom (*Weak Harmonic Transitivity Axiom*)

*If $A \succ B$ and $B \succ C$, then $P\{\{A \succ C\} \oplus \varphi\} \sim P\{A \succ C\}$ almost surely for some periodic event $\varphi$ [Charles-Cadogan, 2017].*

# Representation theorem for abstract HPWF

## Theorem (Random fields of HPWF)

*Let $X$ and $\varphi$ be two disjoint spaces, $\Omega$ be a sample space of cognitive states, $P$ a probability distribution over $X$, $A \subset X$ a closed subset, $x \in A$, and $g_h$ be a continuous function such that $g_h(x) \in \varphi$. In the conjoined space $X \oplus \varphi$, generate an equivalence relation $R$ by $x \sim g_h(x)$ for each $x \in A$. The quotient space $(X \oplus \varphi) / R$ is said to be $X$ attached to $\varphi$ by $g_h$ and is written $X \cup_{g_h} \varphi$ with attaching map $g_h$ [Dugundji, 1966, p. 127]. There exist a mapping $(w \circ P) : X / A \to X \cup_{g_h} \varphi$ into the attached space $X \oplus \varphi$. In particular, the weak harmonic axiom contemplates a harmonic map $g_h(x) \to [0, \epsilon] \subset \varphi$ with the composite mapping $(w \circ P)(x, \omega) = P(x) \oplus g_h(x, \omega)$ where $\{(\Omega, \mathcal{F}, P),\ g_h(x)\}$ is a random field defined on $\varphi$.* □

## Remark

$(w \circ P)(x, \omega) = P(x) \oplus g_h(x, \omega)$ need not be additively separable. However, for analytic tractability one often assumes additive separability. □

# Venture theory representation of PWF

[Hogarth and Einhorn, 1990] descriptive model is a manifestation of our abstract HPWF. It is an *outcome dependent pwf that depends on mental states* and the entire distribution of outcomes.

- $w(p_A) = \overbrace{p_A}^{\text{anchor probability}} + \overbrace{(k_g - k_s)}^{\text{mental states}}$ where DMs *first anchor* on a stated probability $p_A$, *and then adjust* by mentally simulating other values;

- $k_g$ represents weighted values *above* $p_A$; $k_s$ represents weighted values *below* $p_A$

- $k_g = f(\sigma, \theta, p_A, v(x)), \qquad k_s = g(\sigma, \theta, p_A, v(x))$
  $$\frac{\partial k_g}{\partial \sigma} > 0, \quad \frac{\partial k_s}{\partial \sigma} > 0, \qquad \frac{\partial k_g}{\partial \theta} > 0, \quad \frac{\partial k_s}{\partial \theta} > 0$$
  $$\frac{\partial k_g}{\partial p_A} < 0, \quad \frac{\partial k_s}{\partial p_A} > 0, \qquad \frac{\partial k_g}{\partial |v(x)|} > 0, \quad \frac{\partial k_s}{\partial |v(x)|} > 0$$

- $\sigma \triangleq$ measure of uncertainty, i.e., standard deviation of outcomes; $\theta \triangleq$ measure of ambiguity; and $v(x) \triangleq$ [Kahneman and Tversky, 1979] value function. $w(p_A)$ depends on: the entire distribution of outcomes through first and second moments in $\sigma$; and *de facto* mental states $k = (k_g - k_s)$ that fluctuate around $p_A$.

# HPWF Existence Theorem

## Theorem (Harmonic Probability Weighting Function)

*There exist a harmonic representation of probability weighting functions given by*

$$w(\mathbf{x}, p_i) = \overbrace{\eta_0 p_i}^{\text{anchor probability}} + \overbrace{\eta_1 k_n^{(\kappa)}(z_j)}^{\text{mental states}}$$

*where $p_i$ is an anchor probability under EUT, $z_i$ is the Z-score of the payoff associated with $p_i$, $k_n^{(\kappa)}(z_j)$ is the abstract harmonic component (with period parameter $\kappa$) that characterizes DMs' mental states under non-EUT, and $\eta_0$ and $\eta_1$ are* **elevation** *and* **curvature** *parameters, respectively, with identifying restrictions* $\frac{-\eta_0}{\eta_1} \leq k_n^{(\kappa)}(z_j) \leq \frac{1-\eta_0}{\eta_1}$. □

## Remark

We used maximum entropy to show that $\tan(\psi(z))$ is a coherent model of $k_n^{(\kappa)}(z)$ so $w(x, p) = \eta_0 p + \eta_1 \tan(\psi(z))$ is a coherent specification, where $\eta_0$ and $\eta_1$ are **elevation** and **curvature** parameters, resp., and $\psi(z)$ is a phase function with period $\kappa$. It should be noted that $\tan(\psi(z))$ is "not well behaved near the end points" [Kahneman and Tversky, 1979]. So it captures a well know trait of pwfs.

# Representation of mental states in Hilbert space

## Proposition

*Given $z$ is standard normal for a distribution of outcomes $\mathbf{x}$, there exist an abstract harmonic representation of the net adjustment factor $k(z)$ such that*

$$k_n(z) = \sum_{i=1}^{n} k_i^\star \mathbf{r}_i(z)$$

*where $k_i^\star$ is the abstract Fourier coefficient of $k$ such that $\sum_{i=1}^{\infty} |k_i^\star|^2 < \infty$, $z = \sum_{j=1}^{\infty} \epsilon_j(z) 2^{-j}$, and $\mathbf{r}_i(z) = (-1)^{\epsilon_i(z)}$, where $\epsilon_j(z)$ is the $j$-th coefficient in a nonterminating binary expansion of $z$. In particular, if $\sum_{i=1}^{\infty} |k_i^\star|^2 = \sigma_{k_n}^2 < \infty$, then $k_n(z) \sim (0, \sigma_{k_n}^2)$.* □

## Lemma (Mental states. Cf. [Conte et al., 2009])

*Let $\psi(t)$ be the mental state of a DM at time $t$, and $c_j(t) = |k_j^\star / \sigma_{k_n}^2| e^{-i\theta_j t}$. Then (using quantum probability "bra" and "ket" notation) $|\psi(t)\rangle = \sum_j c_j(t) |\phi_j\rangle$ where $\phi_j$ is a basis function for infinite dimensional Hilbert space.* □

Mental state is a superposition of harmonic functions controlled by the Hogarth-Einhorn type adjustment factor.

# Quantum decision theory ruminations

## Theorem (Quantum preference states (de Broglie-Bohm type))

*Let $(X, \mathcal{B}(X), \mu)$ be an outcome measure space, $F$ be an unknown probability distribution over $X$, and $\tilde{w}_m(F)$ be a complex valued functional of $F$. For some $B \in \mathcal{B}(X)$, let $\psi(x)$ represent the state of uncertainty about outcome $x \in B$, and $w(F)$ represent the maximum entropy probability weighting functional (MaxEnt-HPWF) for $F(x)$. Then (suppressing $x$) the state dependent MaxEnt-HPWF is given by*

$$\psi(x) = -\Big\{(1 + \ln(w(F)))w'(F)\Big\}^{-1} \|\tilde{w}_m(F)\|_{L^2_\mu} \exp(-i\vartheta(F))$$

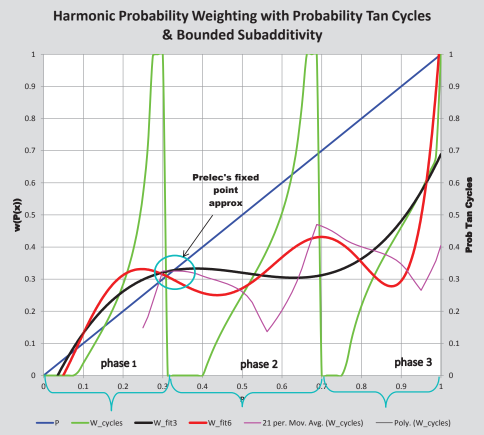
*Furthermore, the probability that uncertain outcome $x$ is in $B$ is given by*

$$P(B) = \frac{1}{\int_X |A(F)|^2 \mu(dF)} \int_B |A(F)|^2 \mu(dF)$$

*where* $A(F) = -\Big\{(1 + \ln(w(F)))w'(F)\Big\}^{-1} \|\tilde{w}_m(F)\|_{L^2_\mu}$. □

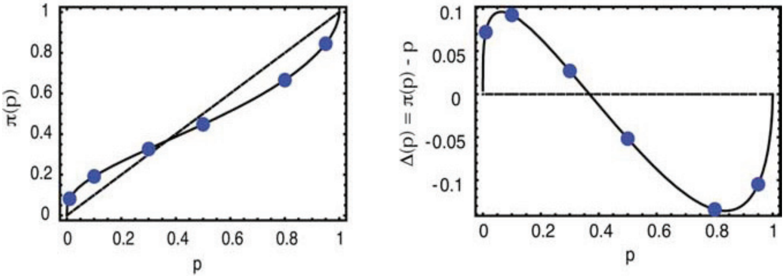
# Applications and implementation of HPWF

# Simulated HPWF with linear phase $\psi(z) = z$



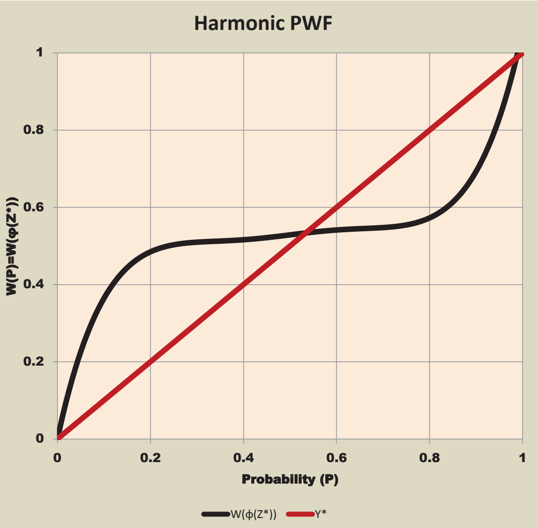

HPWF plot for $w(x,p) = 0.5\,p + 0.2\tan(z)$, where $-4 \leq z \leq 4$ and $0 \leq w(x,p) \leq 1$ for endpoints. $W_{cycles}$ decomposes the probability weighting function space into three latent probability cycles that characterize bounded subadditivity, i.e., **phase 1**: $0 \leq p \leq 0.3$; **phase 2**: $0.3 < p \leq 0.7$; **phase 3**: $0.7 < p \leq 1$. *W_fit3* is a cubic polynomial ($R^2 = 0.216$). Compare [Blavatskyy, 2016] *L*-moments approach. *W_fit6* is a 6-degree polynomial ($R^2 = 0.31$). We also fitted a 21-point moving average to the data.

# Neural Harmonic Probability Weighting




- [Hsu et al., 2009, Fig. 3.A] plots neural pwf (above) from fMRI data. $\alpha$ is estimate using [Prelec, 1998] single parameter pwf $\pi(p, \alpha) = w(p, \alpha)$. $\Delta(p, \alpha) = \pi(p, \alpha) - p$. The pwf is recovered from the specification $\pi(p, \alpha) = a + b_1 p + b_2 \Delta(p, \alpha) + \epsilon$, $-3 \leq b_2 \leq 2$, $0 \leq \alpha \leq 1.6$.
- Under HPWF theory $\eta_0 = b_1$, $\eta_1 = b_2$. We assign $k_n^{(\kappa)}(z_j) = \Delta(p, \alpha)$ as the harmonic component (with period parameter $\kappa$) that characterizes DMs' mental states.
- $\eta_1 = 0.2$ in our simulated HPWF is admissible here since $\eta_1 \in [-3, 2]$. Fitting isomorphism $\tan(\psi(z)) \simeq \Delta(p, \alpha)$ provides a neural phase function or phase response curve. Cf. [Canavier, 2006]

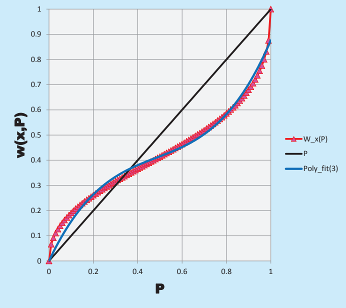
# HPWF from maximum entropy

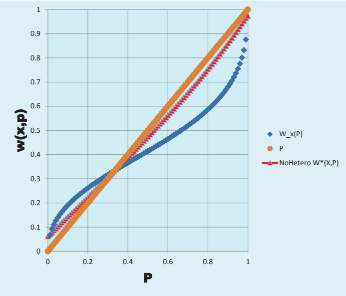



$w(p, x) = 1.5p + \tan(\psi(z))$ where the phase function is

$$\psi(z) = \tan^{-1}\left(\frac{3}{8\sqrt{2\pi}}\tan(\theta(z) - \frac{3}{8\sqrt{2\pi}}z - \frac{7}{24\sqrt{2\pi}}z^3 + \frac{608}{15\sqrt{2\pi}}\right)$$

# Harmonic regression estimates of HPWF




Left plot depicts the calibrated HPWF (red) with no heteroskedasticity correction, and 3rd-degree polynomial fit (blue) of calibrated HPWF. Diagonal is linear probability $w(P) = P$.

$$w(P) = 0.02\,P + 0.001\,\tan(\psi(z)) \qquad \text{(Calibrated)}$$

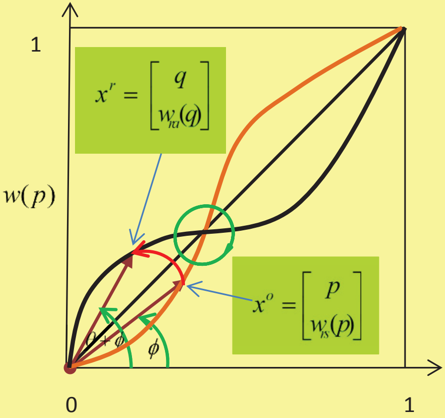
$$\widehat{w(P)} = 2.16\,P^3 - 3.14\,P^2 + 1.86\,P, \quad R^2 = 0.98 \qquad \text{(3rd-poly fit)}$$

Right plot depicts the calibrated HPWF (blue) before heteroskedasticity correction and after correction (red) $w^\star(P) = w(P)/\sqrt{\cos(P)}$ debiasing factor recovers linear probability

$$\widehat{\widehat{w}}^\star(P) = \underset{(0.06)}{0.12}\sqrt{\cos(P)} + \underset{(0.81}{0.02}\left(\frac{\sin(P)}{\sqrt{\cos(P)}}\right), \quad R^2 = 0.9870 \qquad \text{(H-reg*)}$$

# Harmonics from optimism to pessimism

## Behavioural Matrix Operations




### Behavioural motion

$$\mathbf{R}(\theta)\mathbf{A}x^o = x^r, \quad \mathbf{A} = \mathbf{LU}, \quad \dot{x}^o = \mathbf{A}x^o$$

### Rotation

$$\mathbf{R}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

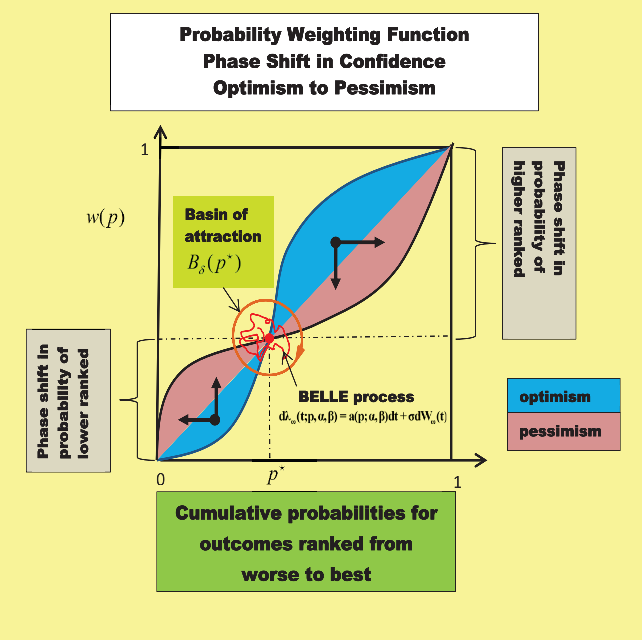
### Translation

$$\mathbf{U} = \begin{bmatrix} \dfrac{q - w_{rs}(p)}{p - w_{rs}(p)} & \dfrac{p - q}{p - w_{rs}(p)} \\ 0 & 1 \end{bmatrix}$$

### Distortion

$$\mathbf{L} = \begin{bmatrix} 1 & 0 \\ \dfrac{w_{rs}(p) - w_{ra}(p)}{w_{rs}(p) - p} & \dfrac{w_{ra}(p) - p}{w_{ra}(p) - p} \end{bmatrix}$$

# Harmonics in phase shifts in confidence




Behavioural Empirical Local Lyapunov Exponent (BELLE) process $d\lambda_n(t,\omega;p,\alpha,\beta) = a(p;\alpha,\beta)dt + \sigma dW_n(t,\omega)$ introduced in [Charles-Cadogan, 2015] characterizes behavioural dynamic system induced by probabilistic risk attitudes. $\lambda_n(t,\omega)$ is empirical local Lyapunov exponent, $\alpha$, $\beta$ is elevation and curvature parameters in [Prelec, 1998] pwf, $\sigma$ is volatility of measurement error, and $W_n(t,\omega)$ is Brownian motion. BELLE process controls the spin of the curves in space around the fix point. It provides closed form expressions for large deviation probabilities of phase shift in investor confidence, and market crash arising from investor uncertainty about market dynamics.

# HPWF resolution of preference reversal puzzle

# Procedure invariant experiments

Let $\boldsymbol{x} = \{x, y, z\}$ be a set of non negative outcomes, $\mathfrak{z}$ be the corresponding Z-scores, $U(\cdot)$ be a utility function that satisfies [Von Neumann and Morgenstern, 1953] axioms, and $\boldsymbol{p} = \{p_x, p_y, p_z\}$ be the corresponding probability distribution of $\boldsymbol{x}$.

## Assumption (Experimental design)

*$E1$ and $E2$ are two temporally spaced experiments that elicit preferences. Subjects induce a ranking of lotteries by choosing preferred lotteries in $E1$, and accept [money] bids to sell each lottery singly in $E2$.* □

## Assumption (Procedure invariance)

*The expected value of a lottery under a choice or willingness to accept (WTA) bid price procedure is the same.* □

## Assumption (Preference reversal)

*In $E1$ our DM expresses the preference $x \succ y \succ z$, and in $E2$ she reverses preference such that $y \succ z \succ x$.* □

# Baseline experiment E1 induce ranking of outcomes

DMs choose lotteries $L \equiv \{L_1, L_2, L_3\}$ in $E1$, and express reservation price (WTA) for each lottery shown separately in $E2$ where:

$$L_1 \equiv (x, p_x;\;\; y, p_y;\;\; 0, 1 - p_x - p_y)$$
$$L_2 \equiv (y, p_y;\;\; z, p_z;\;\; 0, 1 - p_y - p_z)$$
$$L_3 \equiv (x, p_x;\;\; z, p_z;\;\; 0, 1 - p_x - p_z)$$

There are 3! ways in which the lotteries can be ordered.

- Experimental design Proffer a baseline experiment $E1$ with binary choice

$$E1 \sim \{A_s, \theta; \quad L, 1 - \theta\}$$

  where $A_s$ is a payoff obtainable with probability $\theta$, and the [compound] lottery $L$ will be played out if selected with probability $1 - \theta$. If $L$ is chosen, then rational DMs choose: $x \succ y$ in $L_1$; $y \succ z$ in $L_2$; and $x \succ z$ in $L_3$ for transitive preferences $x \succ y \succ z$.
- Choice probability Let $A_v = xp_x + yp_y + zp_z$ be the actuarial value of the lottery $L$. Choose $\theta = \frac{A_v}{A_v + A_s}$ so that it equalizes $A_s$ with the actuarial value of the lottery $L$. That is, $\{A_s, \theta\} \sim \{L, 1 - \theta\}$

# Price bids in second experiment E2 induce ranking

- Scaling and randomization In the second experiment $E2$, the payoffs in $E1$ are scaled by a common factor $c$ and subjects are randomly assigned to any of the following binary choices:

$$E2 \equiv \{cA_s, \theta; \quad c\widehat{L}, 1-\theta\}, \text{ where}$$

$$\widehat{L} \in \Big\{\{L_1, L_2, L_3\},\ \{L_1, L_3, L_2\},\ \{L_2, L_1, L_3\},$$
$$\{L_2, L_1, L_3\},\ \{L_3, L_1, L_2\},\ \{L_3, L_2, L_1\}\Big\}$$

  $c\widehat{L}$ represents the scaled payoffs in $E1$.
- Procedure invariance A constant scale $c$ does not affect transitivity. So $E2$ is equivalent to $E1$ up to randomization.
- Observed PR Assume that for a given $c\widehat{L}$ in $E2$, DMs reverse the choices they made in $E1$. For instance, a DM chooses $A_s$ in $E1$ but has higher reservation price (willingness to accept (WTA) bids) for $c\widehat{L}$ instead of $cA_s$ in $E2$.

# Resolution of PR with harmonic probability weighting

- EUT condition Since $\{\theta, cA_s\} \sim \{1-\theta, c\widehat{L}\}$, under EUT, $\theta U(cA_s) = (1-\theta)U(c\widehat{L})$.
- No PR condition Under Rank Dependent Utility (RDU) generalization of EUT, DMs weigh probabilities with $w(\theta)$ and $(1-w(1-\theta))$ [Quiggin, 1993, p. 57]. No preference reversal in $E2$ requires

$$w(\theta)U(cA_s) = (1-w(1-\theta))U(c\widehat{L})$$

Under HPWF $w(\theta) = \eta_0\theta + \eta_1 \tan(\psi(\mathfrak{z}))$ where $\mathfrak{z}$ is a $Z$-score based on $\boldsymbol{x}$. Substitution for $w(\theta)$ gives

$$\eta_1[\tan(\psi(\mathfrak{z}_{cA_s}))U(cA_s) - \tan(\psi(\mathfrak{z}_{cA_v}))U(c\widehat{L})] = \eta_0[\theta U(cA_s) - (1-\theta)U(c\widehat{L})]$$

Right hand side (RHS) vanishes under EUT condition
- Probability cycles So we are left with phase lock on left hand side (LHS)

$$\tan(\psi(\mathfrak{z}_{cA_s}))U(cA_s) = \tan(\psi(\mathfrak{z}_{cA_v}))U(c\widehat{L}) \Rightarrow \tan(\psi(\mathfrak{z}_{cA_v})) = c_u \tan(\psi(\mathfrak{z}_{cA_s}))$$

where $c_u = U(c\widehat{L})\,/\,U(cA_s)$. The harmonic relationship holds only for complete probability cycles. That is, $\psi(\mathfrak{z}_{cA_s}) = \psi(\mathfrak{z}_{cA_v}) + (2k-1)\pi$, for some $k \in \mathcal{K}$ where $\mathcal{K}$ is the set of all $k$ divisible by some number $d(k)$.
- PR criterion If $k \notin \mathcal{K}$, then equality does not hold and the probability cycle is broken. This manifests as PR in $E2$ for those DMs who chose $A_s$ over $L$ in $E1$, but have higher WTA for $\widehat{c\widehat{L}}$ over $cA_s$ in $E2$ where $L$ is equivalent to $L$ up to randomization and scale $c$.

# PR criterion for valuation of compound lotteries

- Procedure invariance Under the transitivity axiom hypothesis, DMs ordinal selection in $L_1$ should be preserved in $\widehat{L}$. So we can set $c = 1$ without loss of generality.
- Assumed PR DM chooses rank ordered outcomes $x \succ y \succ z$ in $E1$ but express willingness to accept (WTA) bids such that $y \succ z \succ x$ in $E2$
- RDU decision weights Under RDU, decision weights for the rank ordered outcomes in $E1$ and $E2$ are computed in accord with [Quiggin, 1982] transform as follows:

$$\pi_z^1 = w_{E1}(p_z); \quad \pi_y^1 = w_{E1}(p_z + p_y) - w_{E1}(p_z); \quad \pi_x^1 = 1 - w_{E1}(p_z + p_y)$$
$$\pi_x^2 = w_{E2}(p_x); \quad \pi_z^2 = w_{E2}(p_z + p_x) - w_{E2}(p_x); \quad \pi_y^2 = 1 - w_{E2}(p_z + p_x)$$

  $w_{E1}$ and $w_{E2}$ reflect the source of the weighting [Abdellaoui et al., 2011]. For example, $\pi_x^1$ and $\pi_x^2$ are a DM's decision weights for $x$ in $E1$ and $E2$, respectively.
- PR criterion RDU valuation of $E1, E2$ should be the same if there is no PR

$$E1: \quad RDU_1(\boldsymbol{x}, \boldsymbol{p}) = \pi_x^1 U(x) + \pi_y^1 U(y) + \pi_z^1 U(z)$$
$$E2: \quad RDU_2(\boldsymbol{x}, \boldsymbol{p}) = \pi_x^2 U(x) + \pi_y^2 U(y) + \pi_z^2 U(z)$$

- PR implies $RDU_1 \neq RDU_2$ which reduces to

$$(\pi_x^1 - \pi_x^2)U(x) + (\pi_y^1 - \pi_y^2)U(y) + (\pi_z^1 - \pi_z^2)U(z) \neq 0$$

  No PR implies $RDU_1 = RDU_2$ and $\pi_x^1 = \pi_x^2$, $\pi_y^1 = \pi_y^2$, $\pi_z^1 = \pi_z^2$

# Source of PR

- Harmonic decision weights The underlying probabilities $p_x$ and $p_z$ do not change in $E1$ and $E2$. Decision weights are obtained via Quiggin's pwf transform method. Whereupon, HPWF decision weights take the form $\pi_j^{(k)} = \eta_0 p_j + \eta_1 \varphi^{(k)}(z_j)$ where $\varphi^{(k)}$ is harmonic with period $k$ and $z_j$ is the standardized value of outcome $x_j$.

- Ordinal utility condition Since $U$ preserves ordinality, under $E1$ we can normalize $U$ by setting $U(y) = 0$ and $U(x) = 1$ to simplify the analysis. See [Anscombe and Aumann, 1963, p. 201], [Karni and Safra, 1990, p. 493], [Quiggin, 1993, p. 63].

- Reduction of PR criterion after ordinal utility analysis is implemented $(\pi_x^1 - \pi_x^2) + (\pi_z^1 - \pi_z^2)U(z) \neq 0 \Rightarrow -\frac{(\pi_x^1 - \pi_x^2)}{(\pi_z^1 - \pi_z^2)} > 0$ for $U(z) > 0$

- Drivers of PRReduction of PR criterion implies

$$(a): \ (\pi_x^1 < \pi_x^2 \quad \text{and} \quad \pi_z^1 > \pi_z^2) \quad \textbf{or} \quad (b): \ (\pi_x^1 > \pi_x^2 \quad \text{and} \quad \pi_z^1 < \pi_z^2)$$

$$(a): \ \pi_x^1 = \eta_0 p_x + \varphi_{1,x}^{(k)}(\mathfrak{z}_x) < \pi_x^2 = \eta_0 p_x + \varphi_{2,x}^{(k)}(\mathfrak{z}_x) \ \Rightarrow \varphi_{1,x}^{(k)}(\mathfrak{z}_x) < \varphi_{2,x}^{(k)}(\mathfrak{z}_x)$$

$$(b): \ \pi_z^1 = \eta_0 p_z + \varphi_{1,z}^{(k)}(\mathfrak{z}_z) > \pi_z^2 = \eta_0 p_z + \varphi_{2,z}^{(k)}(\mathfrak{z}_z) \ \Rightarrow \varphi_{1,z}^{(k)}(\mathfrak{z}_z) > \varphi_{2,z}^{(k)}(\mathfrak{z}_z)$$

where $\varphi_{1,x}^{(k)}(\mathfrak{z}_x)$, $\varphi_{2,x}^{(k)}(\mathfrak{z}_x)$ and $\varphi_{1,z}^{(k)}(\mathfrak{z}_z)$, $\varphi_{2,z}^{(k)}(\mathfrak{z}_z)$ are the cyclic components of decision weights $\pi$ in $E1$, $E2$, and PR is driven by those components.

# Vanishing preference reversal

- Critical components $\Phi(\tilde{\mathfrak{Z}}_j) = \sum_{k=1}^{j} p_j$,
  $\tilde{\mathfrak{Z}}_j = \Phi^{-1}\Big(\sum_{r=1}^{j} p_r\Big) = \Phi^{-1}\Big(\sum_{r=1}^{j} \Phi(\mathfrak{z}_r)\Big)$, $\boldsymbol{x}_j = (x_1, \ldots, x_j)$ and
  $\boldsymbol{\mathfrak{z}}_j = (\mathfrak{z}_1, \ldots, \mathfrak{z}_j)$ where $\mathfrak{z}_j = (x_j - \bar{\mu}_x)\,/\,\sigma_x$
  $\varphi^{(k)}(\boldsymbol{\mathfrak{z}}_j) = \sin(\Delta\psi^{(k)}(\tilde{\mathfrak{Z}}_j))\sec(\psi^{(k)}(\tilde{\mathfrak{Z}}_j))\sec(\psi^{(k)}(\tilde{\mathfrak{Z}}_{j-1}))$
- Controlling jumps The pairwise cyclical components from $(E1, E2)$ $\Big(\varphi_{1,x}^{(k)}(\mathfrak{Z}_x),\ \varphi_{2,x}^{(k)}(\mathfrak{Z}_x)\Big)$ and $\Big(\varphi_{1,z}^{(k)}(\mathfrak{Z}_z),\ \varphi_{2,z}^{(k)}(\mathfrak{Z}_z)\Big)$ differ by the controlling jump factor $\sin[\Delta\psi_{i,j}^{(k)}(\tilde{\mathfrak{Z}}_j)]$ for $j \in \{x,\ z\}$
- Vanishing PR For sufficiently small jumps
  $\sin[\Delta\psi_{i,j}^{(k)}(\tilde{\mathfrak{Z}}_j)] \approx \Delta\psi_{i,j}^{(k)}(\tilde{\mathfrak{Z}}_j)$ and PR is resolved as $\Delta\psi_{i,j}^{(k)}(\tilde{\mathfrak{Z}}_j) \to 0$

**Theorem (Temporal PR)**

*Preference reversal is due to momentary fluctuations of the evaluative process and it is resolved when probability cycles are complete.* □

# Observer effects on PR

- Anchor probability The experimenter *assigns* an *observed* probability distribution to $\boldsymbol{x}$. Call it $\boldsymbol{p}^o$. In which case we have $p_j^o = p_j + e_j^o$. The DM's anchor probability $p_j$ is unobserved and disturbed by $e_j^o$
- Transitivity axiom implies unobserved decision weights are equal across $E1$ and $E2$. So that $\pi_j^{1^{(k)}} = \pi_j^{2^{(k)}}$
- No observed PR in $E1$, $E2$ imply

$$\pi_j^{o1^{(k)}} = \pi_j^{o2^{(k)}} + \eta_1 \left( \varphi_1^{(k)}(\mathfrak{z}_j^o) - \varphi_2^{(k)}(\mathfrak{z}_j^o) \right)$$

  and that $\varphi_1^{(k)}(\mathfrak{z}_j^o) - \varphi_2^{(k)}(\mathfrak{z}_j^o) = 0$
- Observer effect and source method for decision weights make it more likely that for cyclic components $\varphi_1^{(k)}(\mathfrak{z}_j^o) - \varphi_2^{(k)}(\mathfrak{z}_j^o) \neq 0$.

**Theorem (Observer effect of experimenter misperception of PR)**

*Experimenter ex ante assignment of probabilities to the elements of a statistical ensemble of outcomes interferes with DMs' anchor probability distribution for those outcomes, and induces observed PR when the true state is no PR.* $\square$

# Conclusion

- Axiomatic HPWF is a natural extension of human brain waves.
- HPWF phase function resembles human phase response curve popularized in neuroscience and neurophsiology literature.
- HPWF can be estimated via harmonic regression popular in eye tracking literature
- HPWF introduces probability cycles that characterize the principle of bounded additivity
- HPWF rationalizes quantum probability theory "probability interference factor" with harmonic representation of random fields of mental states.
- Further research includes identification of HPWF and its phase function(s) in laboratory experiments, and in financial markets data. Vizly, does the HPWF explain business cycles? Can it predict mood swings in financial markets? Does a large sample of heterogenous DMs with HPWFs constitute a stable behavioural dynamical system?

# THANK YOU ALL

# References I


Abdellaoui, M., Baillon, A., Placido, L., and Wakker, P. P. (2011).
The Rich Domain of Uncertainty: Source Functions and Their Experimental Implementation.
*American Economic Review*, 101(2):695–723.

Anscombe, F. and Aumann, R. (1963).
A Definition of Subjective Probability.
*Annals of Mathematical Statistics*, 34(1):199–205.

Blavatskyy, P. (2016).
Probability weighting and L-moments.
*European Journal of Operational Research*.
In press. http://dx.doi.org/10.1016/j.ejor.2016.05.007.

Bochner, S. (1955).
*Harmonic Analysis and the Theory of Probability*.
Berkeley, CA: Univ. California Press.

Canavier, C. C. (2006).
Phase Response Curve.
*Scholarpedia*, 1(12):1332.
http://dx.doi.org/10.4249/scholarpedia.1332.

# References II


Cavagnaro, D. R., Pitt, M. A., Gonzalez, R., and Myung, J. I. (2013).
Discriminating among probability weighting functions using adaptive design optimization.
*Journal of Risk and Uncertainty*, 47(3):255–289.
http://dx.doi.org/10.1007/s11166-013-9179-3.

Charles-Cadogan, G. (2015).
Market Instability, Investor Sentiment, And The Probability Weighting Functions Implied By Index Option Prices and Credit Risk.
In Proceedings of 38th Conference on Stochastic Processes and their Applications, World Congress in Probab & Stat, Oxford-Mann Institute, Oxford University.

Charles-Cadogan, G. (2017).
A Weak Harmonic Transitivity Axiom.
Work-in-progress.

Conte, E., Khrennikov, A. Y., Todarello, O., Federici, A., Mendolicchio, L., and Zbilut, J. P. (2009).
Mental States Follow Quantum Mechanics During Perception and Cognition of Ambiguous Figures.
*Open Systems & Information Dynamics*, 16(1):85–100.

Debreu, G. (1958).
Stochastic choice and cardinal utility.
*Econometrica*, 26(3):440–444.

# References III


Dugundji, J. (1966).
*Topology*.
Allyn and Bacon Series in Advanced Mathematics. Boston, MA: Allyn and Bacon, Inc.

Goldstein, W. M. and Einhorn, H. J. (1987).
Expression theory and the preference reversal phenomena.
*Psychological Review*, 94(2):236 – 254.

Hogarth, R. and Einhorn, H. (1990).
Venture Theory: A Model of Decision Weights.
*Management Science*, 36(7):780–803.

Hsu, M., Krajbich, I., and Camerer, C. F. (2009).
Neural response to reward anticipation under risk is nonlinear in probabilities.
*Journal of Neuroscience*, 29(7):2231–2237.

Kahneman, D. and Tversky, A. (1979).
Prospect theory: An analysis of decisions under risk.
*Econometrica*, 47(2):263–291.

Karni, E. and Safra, Z. (1990).
Rank-Dependent Probabilities.
*Economic Journal*, 100(401):pp. 487–495.

# References IV


Lo, A. W., Mamaysky, H., and Wang, J. (2000).
Foundations of Technical Analysis: Computational Algorithms, Statistical Inference, and Empirical Implementation.
*Journal of Finance*, 50(4):1705–1765.

Prelec, D. (1998).
The probability weighting function.
*Econometrica*, 60:497–528.

Quiggin, J. (1982).
A theory of anticipated utility.
*Journal of Economic Behaviour and Organization*, 3(4):323–343.

Quiggin, J. (1993).
*Generalized Expected Utility Theory: The Rank Dependent Model*.
Kluwer Academic Press, Boston, MA.

Tversky, A. (1969).
Intransitivity of Preferences.
*Psychological Review*, 76(1):31–48.

Tversky, A. and Fox, C. R. (1995).
Weighting Risk and Uncertainty.
*Psychological Review*, 102(2):269–283.

# References V


Tversky, A. and Wakker, P. (1995).
Risk Attitudes and Decision Weights.
*Econometrica*, 63(6):1255–1280.

Von Neumann, J. and Morgenstern, O. (1953).
*Theory of Games and Economic Behavior*.
Princeton University Press, 3rd edition.

Wilcox, N. T. (2011).
Comparison of Three Probabilistic Models of Binary Discrete Choice Under Risk.
Work-in-Progress, Economic Science Institute, Chapman University.