Analysis of instabilities in simple mechanical experiments

[L. Renson, JS, DAW Barton, AD Shaw, SA Neild, arxiv:1901.06970]

identified and tracked tipping points
- without tipping
- "without" mathematical model

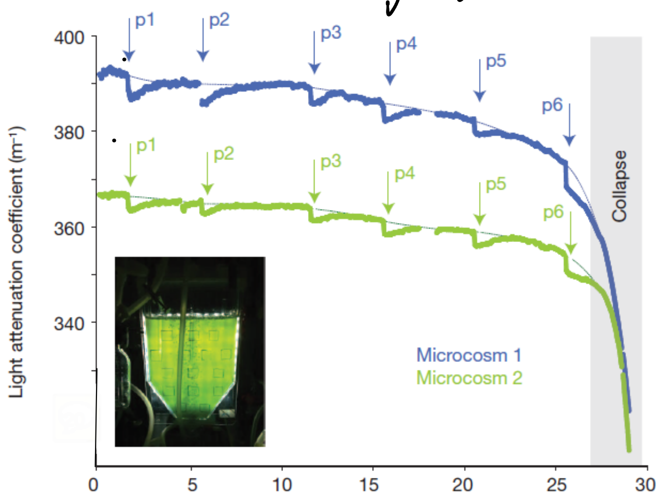
addressed "noise"/uncertainty problems with Gaussian process

tipping point simplest case ~ mathematically passage through a fold

Example from M. Scheffer's group (Veraart et al 2012)

Cyanobacteria under increasing light stress

concentration of c.b. ~



~ light stress]

rough model

A = attenuation (dynamic)

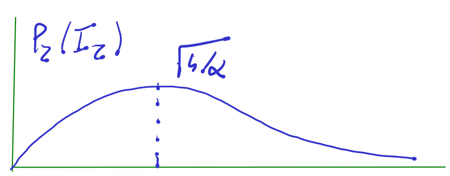
J = light intensity incoming (parameter)

$$\boxed{\frac{dA}{dt} = \underbrace{p_{max} \cdot P(J, A)}_{\text{productivity}} A - l \cdot A =: F(A, J)}$$

$$P(J, A) = \frac{1}{h}\int_0^h p_z(I_z(A, J, z))\,dz$$ average over height

$I_z(A, J, z) = J\exp(-(k_A A + k_d) z)$ light intensity at height z

$p_z(I_z) = \dfrac{I_z}{h_A + I_z + d I_z^2}$ productivity at level z

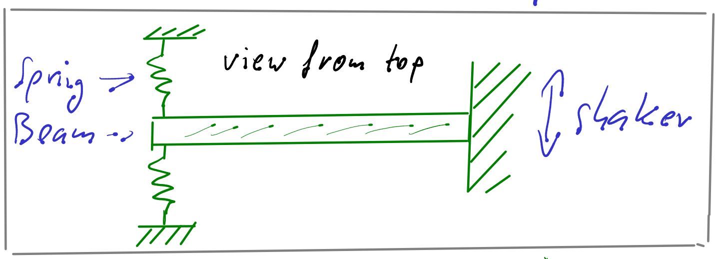


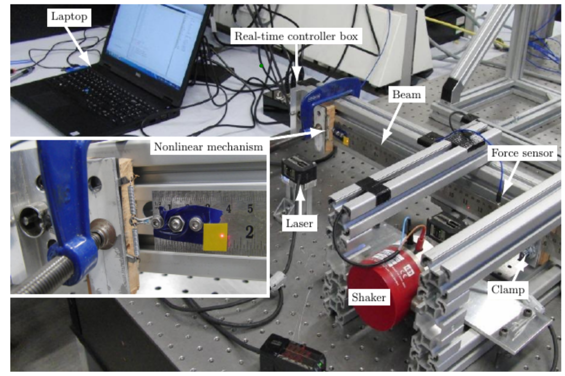
Finding critical value of A, J (= fold) in equation:

$$\boxed{\begin{aligned} 0 &= F(A, J) \\ 0 &= \frac{\partial F}{\partial A}(A, J) = \det \frac{\partial F}{\partial A}(A, J) \end{aligned}}$$

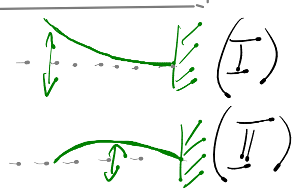
if F is uncertain then $\frac{\partial F}{\partial A}$ is even more uncertain

L Renson et al experiment





Without springs mostly

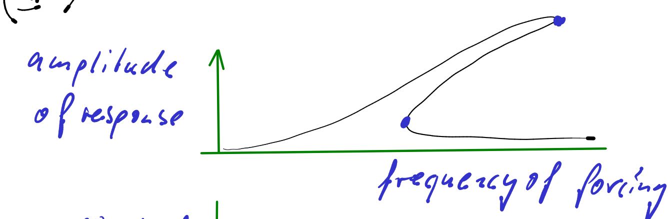


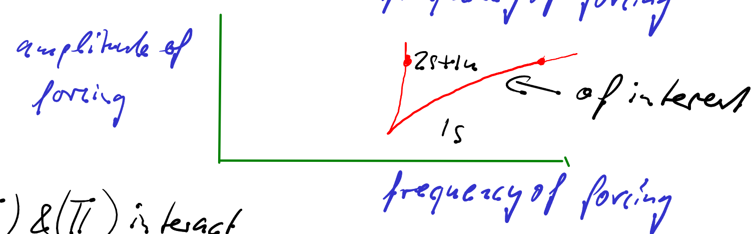
are "in resonance" 1:3

with springs, balanced with

if (I) only occurs, pair of folds

in two parameters





response becomes much more complicated if (I) & (II) interact

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Feedback in Experiment ⇒ Equation

usually, forcing f ⟶ response $x(f)$

Equation

introduce feedback loop: modify forcing to $f = f_{ref} - K\cdot(x - x_{ref})$

control gains

reference (variable in 'equation')

response, depends on f_{ref}, x_{ref}: $x(f_{ref}, x_{ref})$

turns experiment into equation

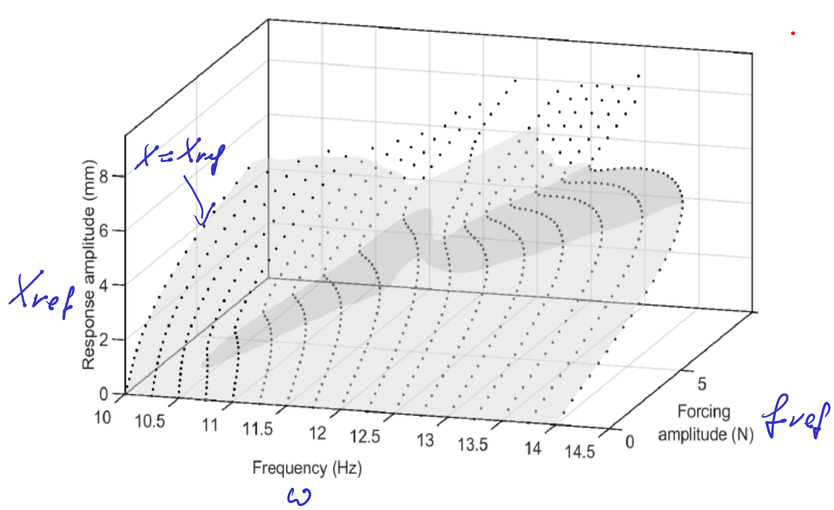
$$\boxed{x(f_{ref}, x_{ref}, \omega) - x_{ref} = 0}$$

high level of noise/uncertainty

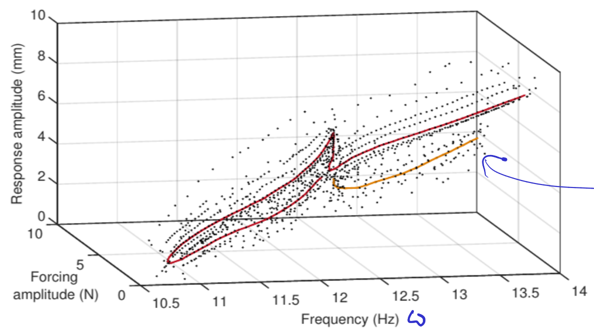
Fold:

$$\boxed{\frac{\partial x}{\partial x_{ref}}(f_{ref}, x_{ref}, \omega) = 1}$$

worse



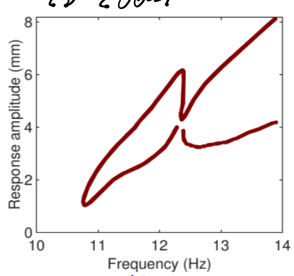
Result



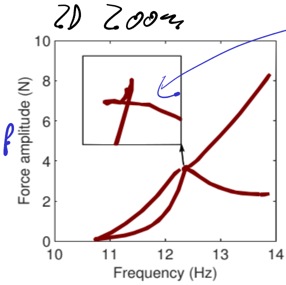
black dot = experiment performed

f_{ref}, ω, x_{ref} are all inputs

2D Zoom



2D Zoom



swallow tail

Standard numerical procedure

Solve equation

$$\underbrace{F(f_{ref}, x_{ref}, \omega)}_{y} := \begin{cases} x(f_{ref}, x_{ref}, \omega) - x_{ref} \\ \frac{\partial x}{\partial x_{ref}}(f_{ref}, x_{ref}, \omega) - 1 \\ + \text{ one more condition} \end{cases}$$

Solve with Newton-Raphson iteration

$$y^{k+1} = y^k - \frac{\partial F}{\partial y}(y^k)\, F(y^k)$$

requires another derivative

[Schilder et al. 2015]

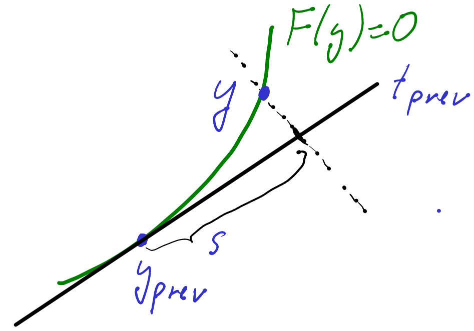
Continuation

Along curve: y_{prev} previous solution

t_{prev} approx. tangent to curve in y_{prev}

extra condition $t_{prev}^T(y - y_{prev}) - s = 0$

stepsize along curve



Gaussian Process = linear interpolation / regression

past measurements at $\hat{y}^1, \dots, \hat{y}^n$: $\hat{x}^1, \dots, \hat{x}^n$, noisy: $\hat{x}^n = \bar{x}^n + \varepsilon_j$, $\varepsilon_j \sim N(0, \sigma_m^2)$

(f, ω, x_{ref}) at n points   responses   normal dist. error

choose correlation function, $\kappa(\hat{y}^i, \hat{y}^j) = \sigma_f^2 \exp\left(- \frac{\| y^i - y^j \|^2}{d^2} \right)$.

covariance matrix $\hat{K}_{ij} = \kappa(\hat{y}^i, \hat{y}^j)$, $j, i = 1 \dots n$ $+ \sigma_m^2$ if $i = j$

mean vector $\hat{x} = (\hat{x}^1 \dots \hat{x}^n)^T$

for arbitrary y we can compute vector $\hat{k}_*(y) = (\kappa(\hat{y}^1, y), \dots, \kappa(\hat{y}^n, y))^T$

number $k_{**}(y) = \kappa(y, y)$

↳ estimate for x at y is: $\text{mean}(x) = \hat{k}_*(y)^T \hat{K}^{-1} \hat{x}$ ← linear in $\hat{x}, y$

$\text{var}(x) = k_{**}(y) - \hat{k}_*(y)^T \hat{K}^{-1} \hat{k}_*(y)$ ← quadratic in y

differentiable functions of input y

$$\boxed{\begin{aligned} m_g(y) &= \hat{k}_*(y)^T \hat{K}^{-1} \hat{x}, \\ \text{var}_g(y) &= k_{**}(y) - \hat{k}_*(y)^T \hat{K}^{-1} \hat{k}_*(y) \end{aligned}}$$

$\hat{x} = (\hat{x}^1, \dots, \hat{x}^n)$ past measurements

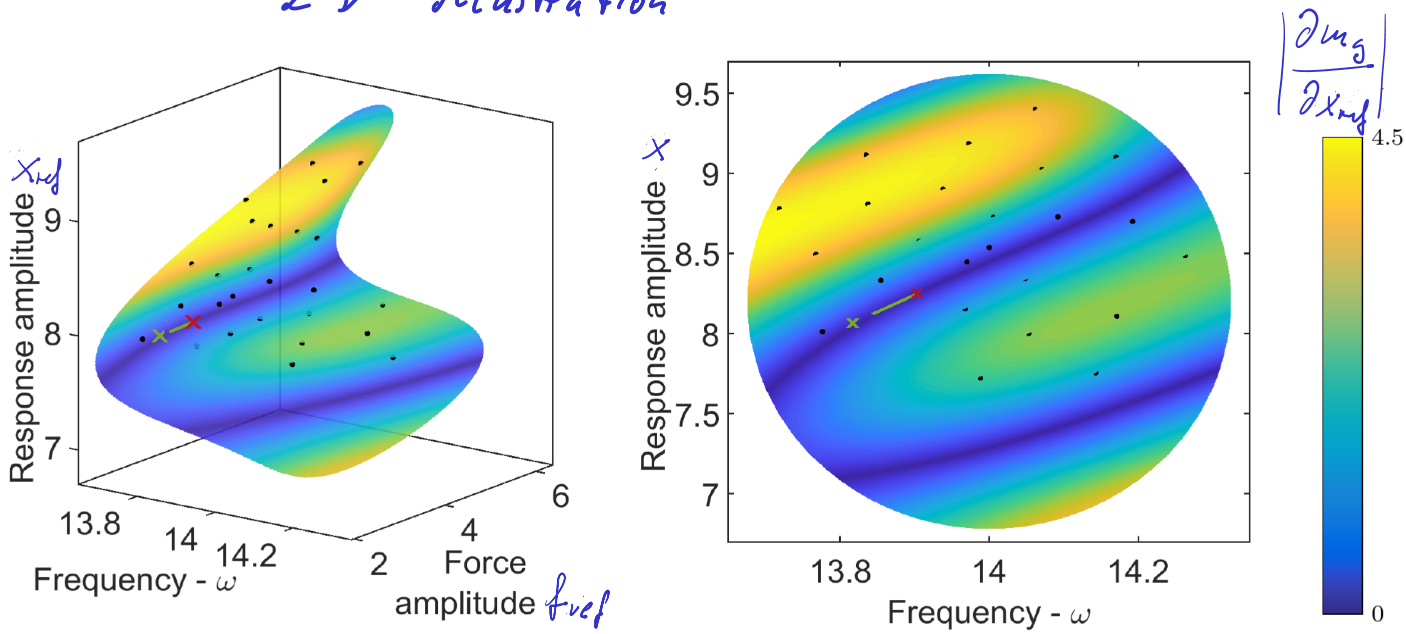
$\underbrace{(\hat{y}^1, \dots, \hat{y}^n)}_{\hat{K}, \hat{k}_*}$ past measurement points

So $m_g(f_{ref}, x_{ref}, \omega)$ is current mean guess of response x at (f, ω, x_{ref}).
$\text{var}_g(f_{ref}, x_{ref}, \omega)$ is uncertainty of this guess.

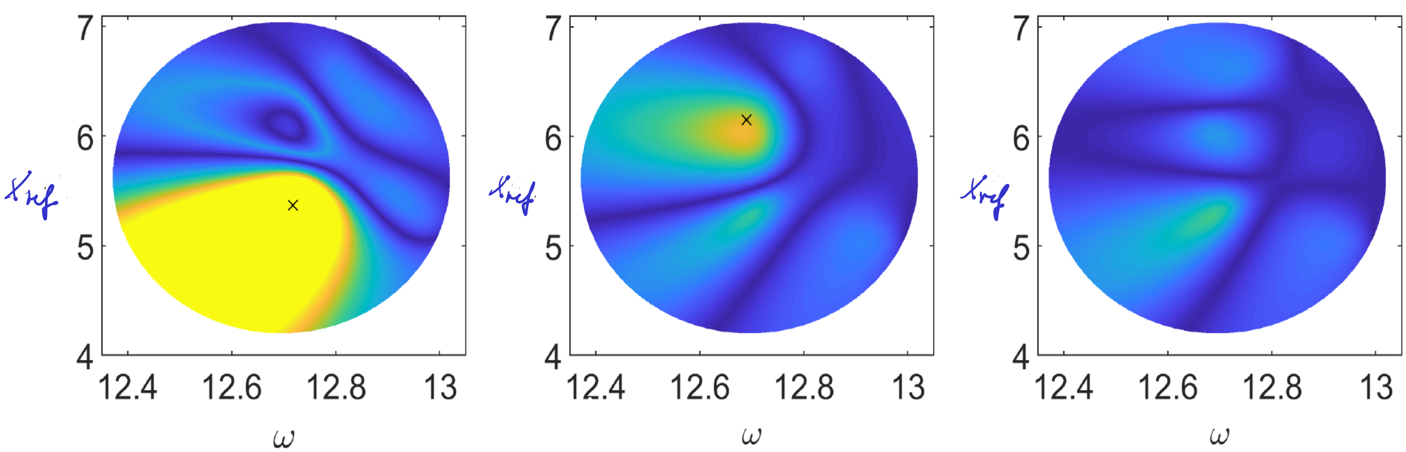
$$\left.\begin{aligned} m_g(f_{ref}, x_{ref}, \omega) - x_{ref} &= 0 \\ \frac{\partial m_g}{\partial x_{ref}}(f_{ref}, x_{ref}, \omega) - 1 &= 0 \end{aligned}\right\} \text{ is easy to solve numerically,}$$

- gives fold curve,
- var_g helps estimate how uncertain fold curve is
- find best point to make new measurement, to reduce uncertainty of fold curve at its boundaries.

2D Illustration

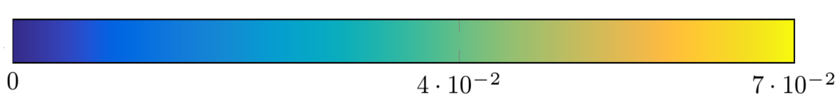


Choosing a new measurement point



(a) (b) (c)

$\left|\frac{\partial \tilde{m}_g}{\partial x_{ref}} - \frac{\partial m_g}{\partial x_{ref}}\right|$



$\tilde{m}_g$ if measurement at new point is taken and equals $m_g(y)+\sqrt{v_{m_g}(y)}$.

at current estimate for fold, $y_e=(f_e, \omega_e, x_{ref,e})$

Conclusion

- finding folds is usually a numerics problem
- large levels of uncertainty make methods from statistics & experimental design more appropriate
- useful for simulations that involve tipping in random or chaotic behaviour