DDEs

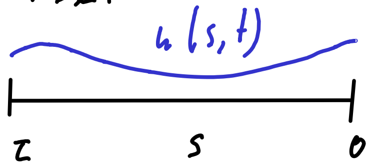
$\dot{x}(t) = f(x(t), x(t-\tau))$,

$x(t) \in \mathbb{R}^n$, $f: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $\tau > 0$

Phase space: cont. fcn on $[-\tau, 0] \to \mathbb{R}^n$

initial value = segment of history

gen. as PDE:



, $x(t) \in \mathbb{R}^n$

$$\boxed{\begin{aligned} \partial_t u(s,t) &= \partial_s u(s,t) \\ u(0,t) &= x(t) \\ \dot{x}(t) &= F(u(\cdot,t)) \end{aligned}}$$

$F: C([-\tau,0];\mathbb{R}^n) \to \mathbb{R}^n$

$F(u) = f(u(0), u(-\tau))$

Linear Problem, eigvals

$\partial_t u = \partial_s u$
$u(0) = x$
$\dot{x} = Au(0) + Bu(-\tau)$

$\lambda u = \partial_s u$

$\Rightarrow u(s) = e^{\lambda s} u(0) = e^{\lambda s} x$

$\lambda x = Ax + Bxe^{-\lambda\tau}$

$\Delta(\lambda) = \lambda I - A - Be^{-\lambda\tau}$ char. matrix

$\dim \Delta(\lambda) < \infty$

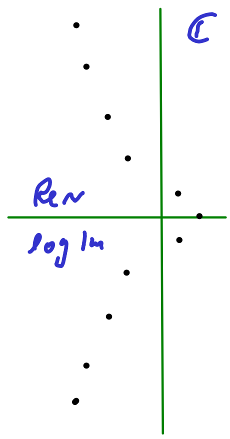
$\det \Delta(\lambda) = 0 \Leftrightarrow \lambda$ Eval

$\Delta(\lambda) v = 0 \Leftrightarrow v$ Evec

alg. mult. of λ

Jordan chain: alg mult as root of $\det \Delta(\lambda)$

$\Delta(\lambda) v_0 = 0$
$\Delta(\lambda) v_1 + \Delta'(\lambda) v_0 = 0$
$\vdots$



Periodic coeffs:

$\dot{x}(t) = A(t)x(t) + B(t)x(t-\tau)$

$A(t) = A(t-1)$, $B(t) = B(t-1)$, $\tau < 1$

char. matrix $\Delta(\lambda)$ for Floquet m.

ODE: $\dot{x}(t) = A(t)x(t)$

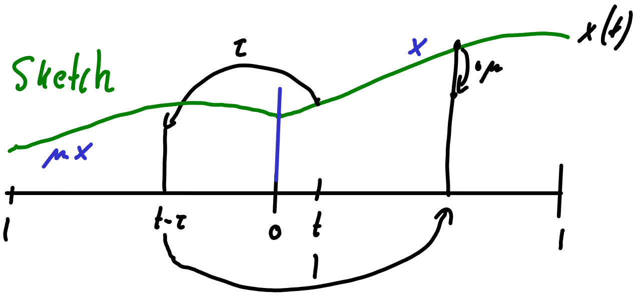
$\Delta(\lambda) = \lambda I - M$ where $Mv = x(1)$

& x solves IVP $\dot{x}(t) = A(t)x(t)$
$x(0) = v$

PDE: $\Delta(\mu)$ where μ^{-1} is F. m., otherwise Δ has ess. sing. at 0

BVP
$$\boxed{\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)\begin{cases} x(t-\tau) & \text{if } t \geq \tau \\ \mu x(t+1-\tau) & \text{if } t < \tau \end{cases} \\ x(0) &= \mu x(1) \end{aligned}}$$

Sketch



[R. Szalai] $\Delta(\mu) = I - \mu M(\mu)$ where $M(\mu) v = x(1)$ & x solves IVP

IVP
$$\boxed{\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)\begin{cases} x(t-\tau) & \text{if } t \geq \tau \\ \mu x(t+1-\tau) & \text{if } t < \tau \end{cases} \\ x(0) &= v \end{aligned}}$$

? unique sol x on $[0,1]$ for all μ ?

$\Delta(\mu) v = 0 \Leftrightarrow$

$v = \mu M(\mu) v \Leftrightarrow$

$x(0) = \mu x(1)$

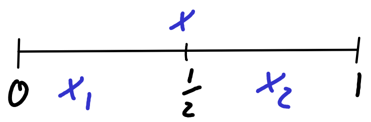
Obs: • $\mu = 0$ ✓ $\rightarrow$ $\Delta(\mu)$ has at most finitely many poles $|\mu| \le R$

• $\Delta(\mu)$ is char. matrix [Knut: $\Delta(-1)v = 0$, $\|v\| = 1$ for PD of p.o.]

• req. for unique sol:
$$\dot{x} = A(t)x + B(t)\begin{cases} x(t-\tau) & \text{if } t \ge \tau \\ \mu x(t+1-\tau) & \text{if } t \le \tau \end{cases}, \quad x(0)=0$$
has only sol $x(t)=0$

• counter example:
$$\dot{x} = x(t - \tfrac{1}{2}) \text{ on } [0,1] \Rightarrow \dot{x} = \begin{cases} x(t-\frac{1}{2}) & \text{if } t \ge \frac{1}{2} \\ \mu x(t+\frac{1}{2}) & \text{if } t < \frac{1}{2} \end{cases}, \quad x(0)=0$$

$$(\Leftrightarrow) \quad \boxed{\begin{array}{ll} \dot{x}_2 = x_1 & , x_1(0) = 0 \\ \dot{x}_1 = \mu x_2 & , x_2(\frac{1}{2}) = x_1(\frac{1}{2}) \end{array}}$$

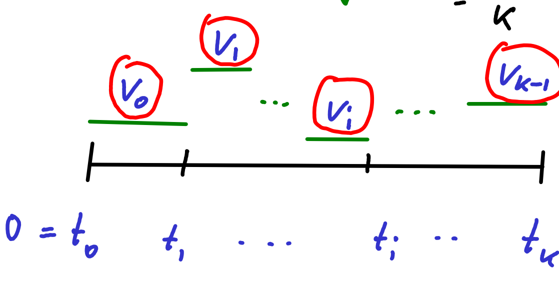


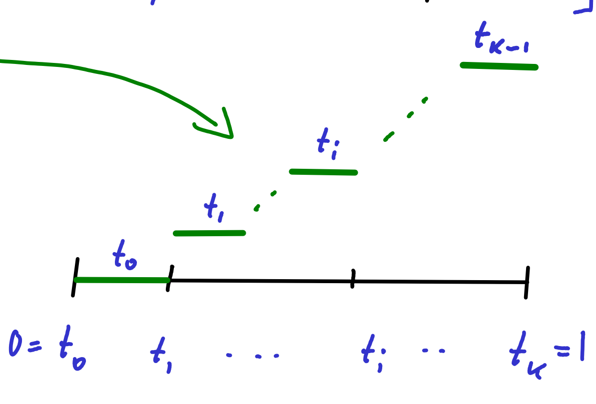
has non-trivial solution x if $0 = 1 - \sqrt{\mu}\sinh\left(\frac{\sqrt{\mu}}{2}\right)$

• Resolution push poles outside of radius R

$$\text{IVP} \Leftrightarrow x(t) = v + \underbrace{\int_0^t}_{\le 1} \underbrace{A(s)}_{\|\ \| < a} x(s) + \underbrace{B(s)}_{\|\ \| \le B_1} \left\{ \begin{array}{ll} x(s-\tau) & \text{if } s \ge \tau \\ \underbrace{\mu}_{\le R} x(s+1-\tau) & \text{if } s < \tau \end{array} \right\} ds$$

$$x(t) = v_s(t) + \int_{\underbrace{a_s(t)}_{\le \frac{1}{k}}}^{t} \left[A(s)x(s) + B(s)\begin{cases} x(s-\tau) & \text{if } s \ge \tau \\ \mu x(1+s-\tau) & \text{if } s < \tau \end{cases} \right] ds$$





$$x = v + L_1 x + \mu L_2 x$$

$$[L_1 x](t) = \int_{a_s(t)}^{t} A(s)x(s) + B(s)\{x(t-\tau) \text{ if } t \ge \tau\} ds$$

$$\|L_1 x\|_\infty \le \frac{1}{k}[a + B] |x|$$

$$[L_2 x](t) = \int_{a_2(t)}^{t} B(s) \begin{cases} 0 \\ x(t+1-\tau) & \text{if } t<\tau \end{cases} ds$$

$$\|L_2 x\| \le |\mu| \frac{b}{k}$$

↳ if $|\mu| \frac{b}{k} + \frac{[a+b]}{k} < 1$ fixed point problem

has unique sol $x(t)$ dep. linearly on $v_0, \dots, v_{k-1}$, analyt. on μ for

$$|\mu| < \frac{k - [a+b]}{b} .$$

$\Delta(\mu)$ is $nk \times nk$ matrix: $\Delta(\mu) \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{k-1} \end{bmatrix} = \begin{bmatrix} \mu x(1) - v_0 \\ \lim_{t \nearrow t_i} x(t) - v_i \end{bmatrix}_{i=1..k-1}$

- this is not a discretization / approximation
- dim $\Delta(\mu)$ depends on $\|A\|, \|B\| \Rightarrow$ on period
  also on $|\mu|$
- Proof of properties (~ [R. Sz., K & VL])

$$\begin{bmatrix} \Delta(\mu) & 0 \\ 0 & I \end{bmatrix} = F(\mu) \cdot [I - \mu \cdot T] \cdot E(\mu)$$

time-1 map of DDE

inf. dim.

isomorphisms

Application: spectrum for large delay

Observation (S. Yanchuk): assume $\dot{x} = f(x(t), x(t-\tau_0))$ has regular periodic orbit γ of period T, and $\left.\frac{\partial T}{\partial \tau}\right|_{\tau=\tau_0} \neq 0$

period reappears at

$$\tau_N = \tau_0 + N T(\tau_0)$$

$$\left.\frac{\partial \tau_N}{\partial \tau}\right|_{\tau=\tau_0} = 1 + N \left.\frac{\partial T}{\partial \tau}\right|_{\tau=\tau_0}$$ → Branches stretch and overlap for large N

↳ for every sufficiently large τ there ex. $\sim \left[\frac{\tau}{\tau_0} \cdot \left.\frac{\partial T}{\partial \tau}\right|_{\tau=\tau_0}\right]$ periodic orbits of shape $\approx \gamma$

Criterion for stability of $\approx y$ for large τ (DDE-Biftool's matrices grow in size $\sim \tau$)

Eigenvalue problem in Floquet exponent form:

$\exp(\lambda)$ is Floquet multiplier $\Leftrightarrow$ (Period $T=1$)

$$\dot y = [A(t)-\lambda I]\,y(t) + \exp(-\tau\lambda)\,B(t)\,y((t-\tau) \cancel{\bmod 1})$$

$$y(0)=y(1) \qquad \exp((-\tau_0-N)\lambda)\,B(t)\,y((t-\tau_0)\cancel{\bmod 1})$$

$\tau = \tau_0 + N$ — large

assume $\mathrm{Re}\,\lambda > c > 0$: $\Rightarrow$ for $N\to\infty$ $\exp((-\tau_0-N)\lambda)\to 0$ $\Rightarrow$ $\dot y = [A(t)-\lambda I]\,y(t)$

strongly unstable spectrum (finitely many)

$$\lambda = \frac{\gamma}{N+\tau_0} + i\omega$$

$$\Rightarrow \dot y = [A(t) - (i\omega + \underset{\downarrow\,0}{\tfrac{\gamma}{N+\tau}})I]\,y(t) + \exp[-\gamma - i\underbrace{\omega(N+\tau_0)}_{\varphi}]\,B(t)\,y(t-\tau_0)$$

$$\dot y = [A(t) - i\omega I]\,y(t) + \exp(-\gamma - i\varphi)\,B(t)\,y(t-\tau_0)$$

solutions define curves $(\gamma + i\omega, \varphi) \in \mathbb{C} \times [-\pi,\pi]$

characteristic fcn:

$$h(\lambda, \exp(-(N+\tau_0)\lambda)) = \det(\Delta(\lambda, \exp((-\tau_0-N)\lambda))) = 0$$ $\leftarrow$ roots λ are Floquet exp.

roots of $h(\lambda, 0)$ are strongly continuous spec. if $\mathrm{Re}\,\lambda > 0$

roots of $h\left(i\omega + \frac{\gamma}{N+\tau_0}, \exp(-\gamma - i\omega(N+\tau))\right)$ are rescaled Floquet exponents

root curves of $h(i\omega, \exp(-\gamma - i\varphi))$ are asympt. cont. spec.

curves can be written as $\gamma(\omega)$, $\varphi(\omega)$ if $h(i\omega, 0) \neq 0$

Floquet exponents on the curve given by

$$\omega_k = \frac{2k\pi}{N+\tau_0} + \frac{1}{N+\tau_0}\varphi\left(\frac{2k\pi}{N+\tau_0}\right) + O((N+\tau_0)^{-2})$$

$$\gamma_k = \gamma(\omega_k) + O((N+\tau_0)^{-1})$$

One curve $\gamma(\omega)$ always touches imaginary axis: curves easy to compute