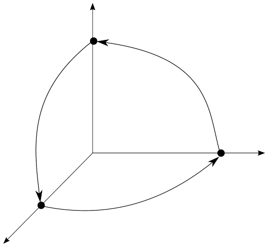
Existence and stability of elementary heteroclinic networks in $\mathbb{R}^4$

Alexander Lohse – Department of Mathematics, Hamburg



Joint work with
Sofia Castro (University of Porto)

Dynamics Reading Group Exeter – 26th February 2015

Definition 1

Let $\Gamma \subset O(n)$ be a finite group and consider an equivariant ode on $\mathbb{R}^n$

$$\dot{x} = f(x) \qquad \text{with} \qquad f(\gamma x) = \gamma f(x) \text{ for all } \gamma \in \Gamma, x \in \mathbb{R}^n.$$

A **heteroclinic cycle** is a collection of finitely many equilibria ξ_j and connecting trajectories $s_j \subset W^u(\xi_j) \cap W^s(\xi_{j+1})$, where $\xi_{m+1} = \xi_1$.

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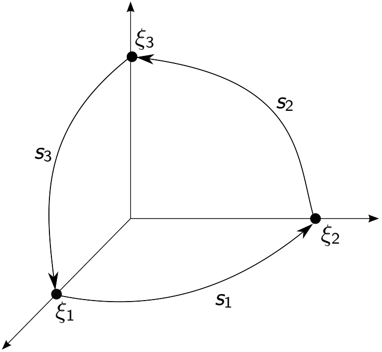
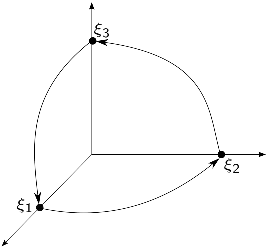


Definition 2 (Krupa&Melbourne [1])

(1) A heteroclinic cycle is called **robust** if for all j there is a subgroup $\Sigma_j \subset \Gamma$ such that ξ_{j+1} is a sink in $P_j := \mathrm{Fix}(\Sigma_j)$ and $W^u(\xi_j) \cap P_j \subset W^s(\xi_{j+1})$.

(2) A robust cycle in $\mathbb{R}^4$ is called **simple** if
- $\dim(P_j) = 2$ for all j,
- it intersects connected components of $(P_{j-1} \cap P_j) \setminus \{0\}$ at most once,
- the linearisation $df(\xi_j)$ has no double eigenvalues.



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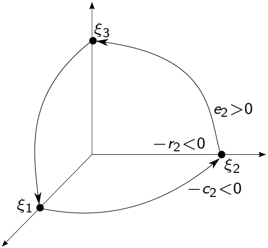

Transverse eigenvalues ("away from the cycle") $t_j \gtrless 0$ influence stability.

Definition 3 (Krupa&Melbourne [1])

A simple heteroclinic cycle $X \subset \mathbb{R}^4$ is of

- **type A** if and only if $\Sigma_j \cong \mathbb{Z}_2$ for all j,
- **type B** if and only if X lies in a 3d fixed-point subspace $Q \subset \mathbb{R}^4$,
- **type C** if and only if it is not of type A or B.

Lemma 4 (Krupa&Melbourne [1])

A simple heteroclinic cycle $X \subset \mathbb{R}^4$ is of type A if and only if there is no element $\gamma \in \Gamma$ that acts as a reflection on $\mathbb{R}^4$.

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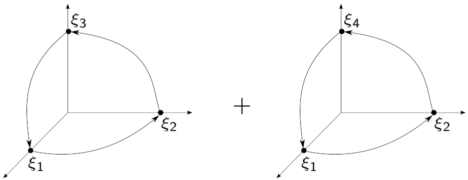
Lemma 5 (Krupa&Melbourne [1])

There are seven simple heteroclinic cycles of types B and C in $\mathbb{R}^4$ and the only finite groups $\Gamma \subset O(n)$ allowing them are the ones denoted in parentheses:

- $B_1^+(\mathbb{Z}_2 \ltimes \mathbb{Z}_2^3)$, $B_2^+(\mathbb{Z}_2^3)$, $B_1^-(\mathbb{Z}_3 \ltimes \mathbb{Z}_2^4)$, $B_3^-(\mathbb{Z}_2^4)$
- $C_1^-(\mathbb{Z}_4 \ltimes \mathbb{Z}_2^4)$, $C_2^-(\mathbb{Z}_2 \ltimes \mathbb{Z}_2^4)$, $C_4^-(\mathbb{Z}_2^4)$

Elementary networks

A **heteroclinic network** is a connected union of finitely many cycles:



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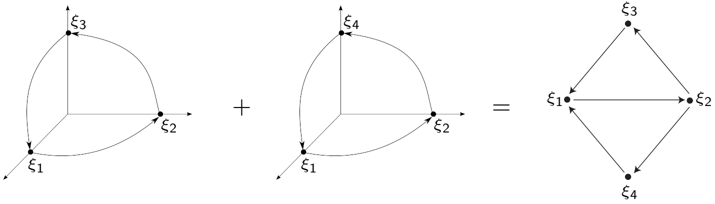

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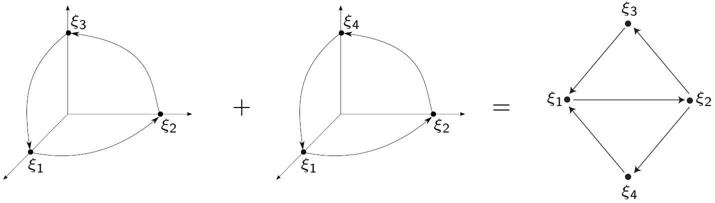

Definition 6 (L.&Castro [3])

A network is called **elementary** if

(a) all of its subcycles are simple,
(b) all of its connections are genuinely heteroclinic,
(c) there are no critical elements other than the network and the origin.

Lemma 7 (L.&Castro [3])

In $\mathbb{R}^4$, the following is the complete list of elementary heteroclinic networks:

- (A_2, A_2), (A_3, A_3), (A_3, A_4), (A_3, A_3, A_4)
- (B_2^+, B_2^+), (B_3^-, B_3^-)
- (B_3^-, C_4^-), (B_3^-, B_3^-, C_4^-)

Definition 8 (Podvigina&Ashwin [4])

A compact invariant set $X \subset \mathbb{R}^n$ is called **predominantly asymptotically stable (p.a.s.)** if it is asymptotically stable relative to a set $N \subset \mathbb{R}^n$ and

$$\frac{\ell(B_\varepsilon(X) \cap N)}{\ell(B_\varepsilon(X))} \xrightarrow{\varepsilon \to 0} 1.$$

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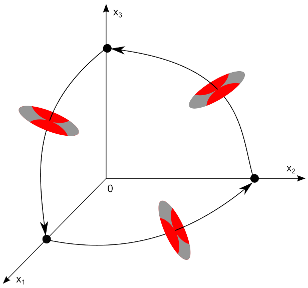

Definition 9 (Podvigina&Ashwin [4])

Let $X \subset \mathbb{R}^n$ be a compact, invariant set. Denote by $\mathcal{B}(X)$ its basin of attraction and for $\varepsilon > 0$ by $B_\varepsilon(x)$ an ε-neighbourhood of $x \in X$. Then set

$$\Sigma_\varepsilon(x) := \frac{\ell(B_\varepsilon(x) \cap \mathcal{B}(X))}{\ell(B_\varepsilon(x))}.$$

Define the **stability index** at x as $\sigma(x) := \sigma_+(x) - \sigma_-(x)$ where

$$\sigma_-(x) := \lim_{\varepsilon \to 0} \frac{\ln(\Sigma_\varepsilon(x))}{\ln(\varepsilon)} \quad \text{and} \quad \sigma_+(x) := \lim_{\varepsilon \to 0} \frac{\ln(1 - \Sigma_\varepsilon(x))}{\ln(\varepsilon)}.$$

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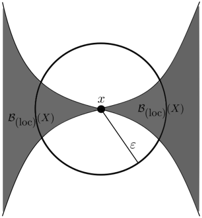

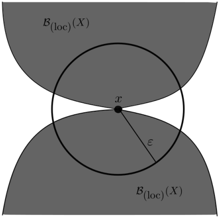

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The **local stability index** $\sigma_{\mathrm{loc}}(x)$ is defined analogously by replacing $\Sigma_\varepsilon(x)$ with

$$\Sigma_{\varepsilon,\delta}(x) := \frac{\ell(B_\varepsilon(x) \cap \mathcal{B}_\delta(X))}{\ell(B_\varepsilon(x))}.$$

Theorem 10 (Podvigina&Ashwin [4])

The stability index $\sigma_{(\text{loc})}(x)$ is constant along trajectories.

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Theorem 11 (L. [2])

Let $X \subset \mathbb{R}^n$ be a heteroclinic cycle with $\ell_1(X) < \infty$. Assume that the local stability index $\sigma_{\mathrm{loc}}(x)$ exists for all $x \in X$. Then the following holds:

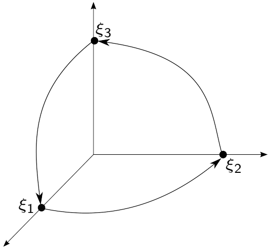
$$(a)\ \ X \text{ is p.a.s. } \Leftrightarrow\ \sigma_{\mathrm{loc}}(x) > 0 \text{ along all connections}$$

Moreover, if X is isolated we also have:

$$(b)\ \ X \text{ is p.u. } \Leftrightarrow\ \sigma_{\mathrm{loc}}(x) < 0 \text{ along all connections}$$

B_3^-- and A_3-cycles

B_3^-- and A_3-cycles are geometrically identical.



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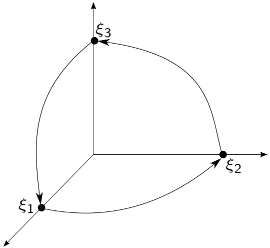
Symmetries $\kappa_i, \kappa_{ij}, \kappa_{ijk} : \mathbb{R}^4 \to \mathbb{R}^4$

$$\kappa_1(x_1, x_2, x_3, x_4) = (x_1, -x_2, -x_3, -x_4)$$



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symmetry Γ	$\langle \kappa_1, \kappa_2, \kappa_3, \kappa_4 \rangle \cong \mathbb{Z}_2^4$	$\langle \kappa_{12}, \kappa_{23}, \kappa_{34} \rangle \cong \mathbb{Z}_2^3$
isotropy spaces	lines ($\mathbb{Z}_2^3$), planes ($\mathbb{Z}_2^2$), spheres ($\mathbb{Z}_2$)	lines ($\mathbb{Z}_2^2$), planes ($\mathbb{Z}_2$)
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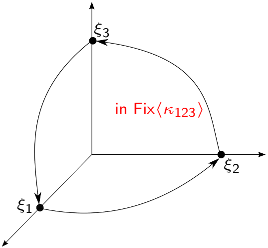

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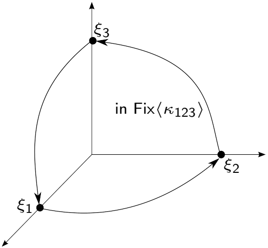

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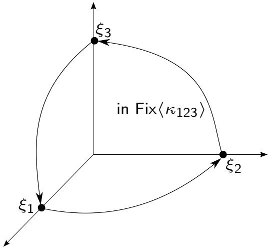

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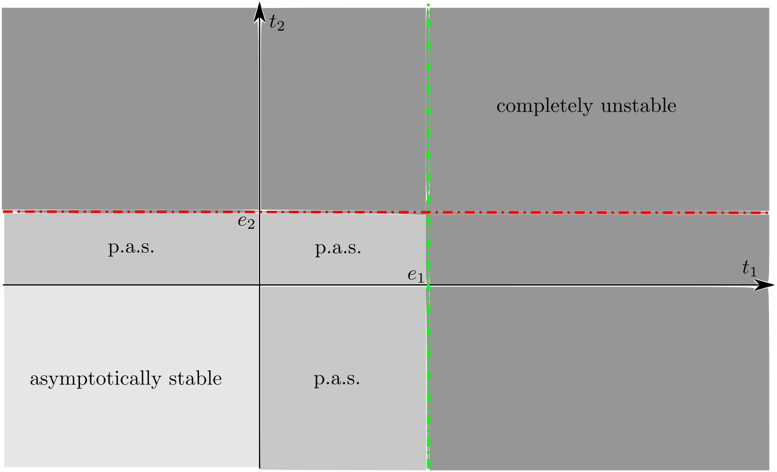
$\to$ How much do the stability properties of these cycles differ?

$$\dot{x}_j = a_j x_j + \left(\sum_{i=1}^{4} b_{1i} x_i^2\right) x_j + c_j x_1 x_2 x_3 x_4 x_j$$

B_3^-- and A_3-cycles – stability



A_3-cycles, $t_3 < 0$

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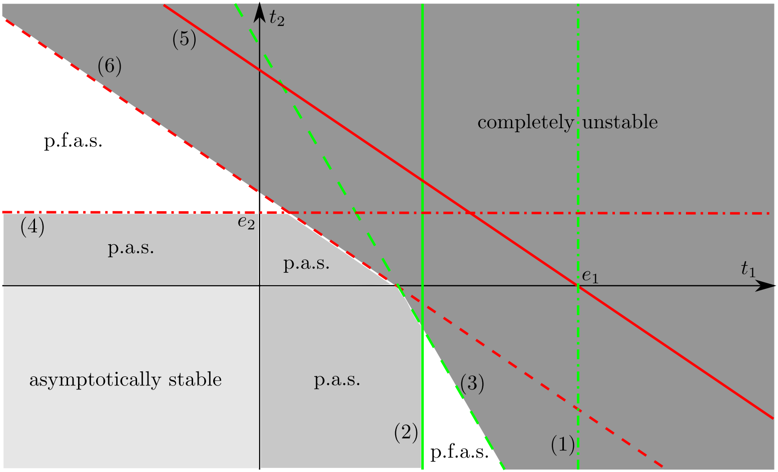

B_3^--cycles, $-c_3 < t_3 < 0$

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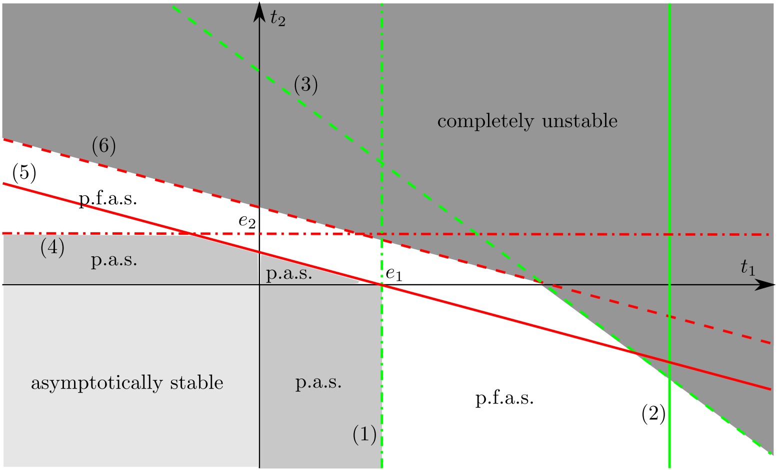
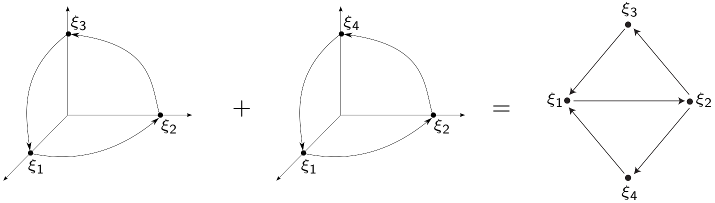


B_3^--cycles, $t_3 < -c_3 < 0$

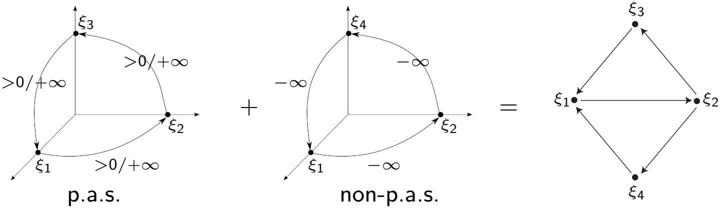
(B_3^-, B_3^-)- and (A_3, A_3)-networks



The corresponding networks are also geometrically identical ...

. . . but their stability properties differ:

(A_3, A_3)



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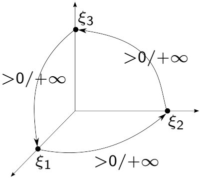
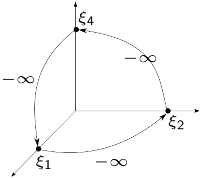


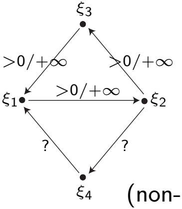
p.a.s.

$+$



non-p.a.s.

$=$



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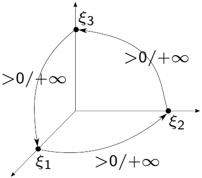

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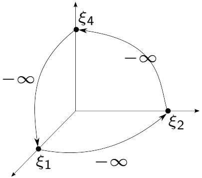

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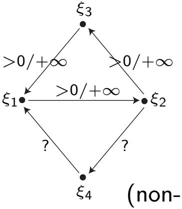
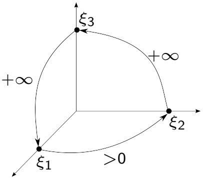


(non-)p.a.s.?

(B_3^-, B_3^-)
(1a)



p.a.s.

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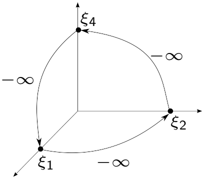

non-p.a.s.

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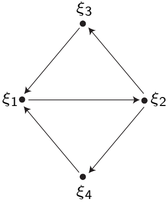

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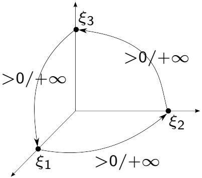
(A_3, A_3)



p.a.s.

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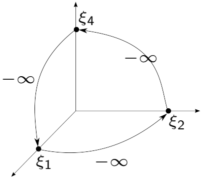

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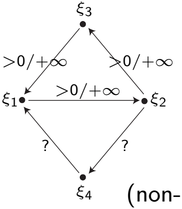

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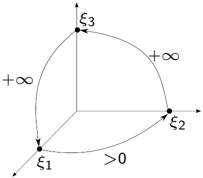

p.a.s.

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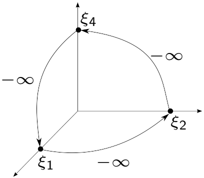

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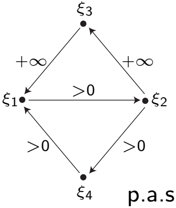

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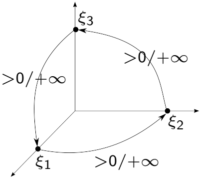
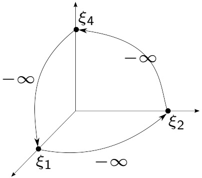
. . . but their stability properties differ:

(A_3, A_3)



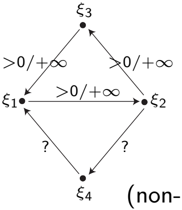
p.a.s.

$+$



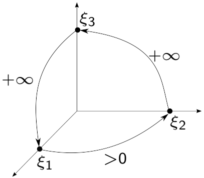
non-p.a.s.

$=$



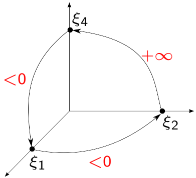
(non-)p.a.s.?

(B_3^-, B_3^-)
(1b)



p.a.s.

$+$



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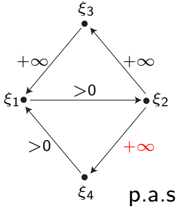
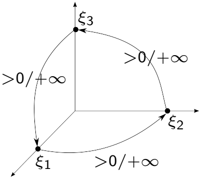
$=$



p.a.s.

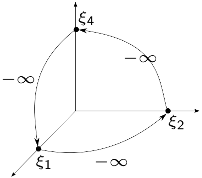
…but their stability properties differ:

(A_3, A_3)



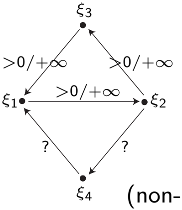
p.a.s.

$+$



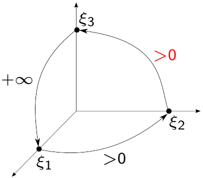
non-p.a.s.

$=$



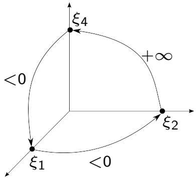
(non-)p.a.s.?

(B_3^-, B_3^-) (2)



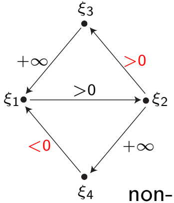
p.a.s.

$+$



non-p.a.s.

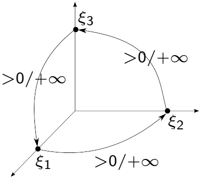
$=$



non-p.a.s.

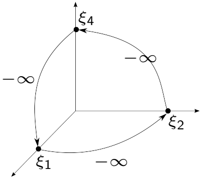
. . . but their stability properties differ:

(A_3, A_3)



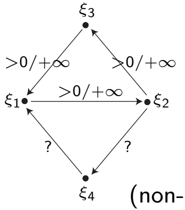
p.a.s.

+



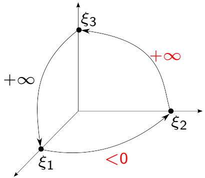
non-p.a.s.

=



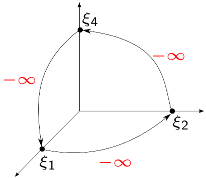
(non-)p.a.s.?

(B_3^-, B_3^-)
(3)



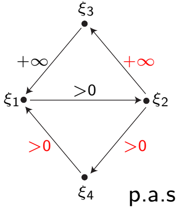
non-p.a.s.

+



non-p.a.s.

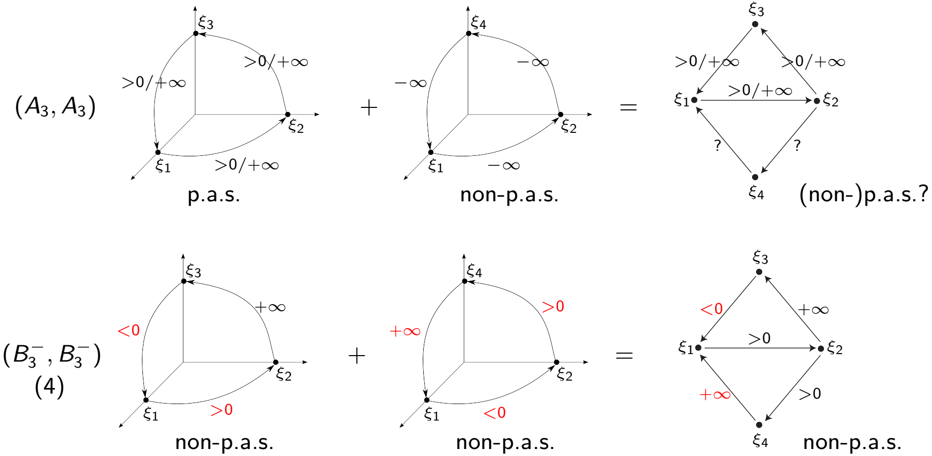
=



p.a.s.

(B_3^-, B_3^-)- and (A_3, A_3)-networks

... but their stability properties differ:



Summary

- Elementary networks are built from simple cycles in the simplest way imaginable.
- There are eight elementary heteroclinic networks in $\mathbb{R}^4$.
- They display complex forms of non-asymptotic stability depending on the symmetry group Γ:
  - less symmetry (type A) – uniform stability along connections, all indices have the same sign
  - more symmetry (type B) – varying stability configurations, indices with different sign possible

→ This is not apparent from their geometry.

Literature

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Thank you very much for your attention.