(Ω, X, P)

sample space: Ω; RV: $X : \Omega \to \mathbb{R}$; Probability: P

$\{X \leq u\} = \{\omega \in \Omega : X(\omega) \leq u\}$.

$\omega \in \Omega$ (sample point).

Let $M_n = \mathrm{Max}\{X_1, X_2, \dots X_n\}$.

$\{X_i\}$ a sequence of IID RVs.

$\{M_n \leq u\} \leftrightarrow \{\omega \in \Omega : X_1(\omega) \leq u, X_2(\omega) \leq u \cdots X_n(\omega) \leq u\}$.

X_i IID $\Rightarrow$

$$P\{M_n \leq u\} = P(X_1 \leq u) P(X_2 \leq u) \cdots P(X_n \leq u)$$
$$= P(X \leq u)^n.$$

This leads to:

<u>Proposition</u>: Suppose $\{X_i\}$ are IID, and suppose $\exists\, a_n, b_n$ and a smooth function $\tau : \mathbb{R} \to \mathbb{R}$ such that $n P(X > \frac{u}{a_n} + b_n) \to \tau(u)$ as $n \to \infty$.

Then $\lim_{n \to \infty} P(M_n \leq a_n) = e^{-\tau(u)}$.

eg $X_i \sim \mathrm{Exp}(1)$: $f_X(x) = e^{-x}$

$P(X_1 > \frac{u}{a_n} + b_n) = \left(1 - e^{-\frac{u}{a_n}} e^{-b_n}\right)$.

If $a_n = 1, b_n = \log n$. Then $P(M_n \leq u_n) \to e^{-\tau(u)}$,

$\tau(u) = e^{-u}$.

(Gumbel).

Extremes for dynamical systems

(M, f, μ, ϕ). M: Manifold $\subseteq \mathbb{R}^d$

$f: M \to M$ discrete map, $x_0 \in M$, $x_n = f^n(x_0)$

μ: ergodic, invariant measure:

$\mu(f^{-1}A) = \mu(A)$, and if $f^{-1}A = A$ then $\mu(A) \in \{0,1\}$.

In particular $\forall A \subset M$, $\frac{1}{n} \sum_{k=0}^{n-1} \mathbb{1}_A(f^k x) \to \mu(A)$.

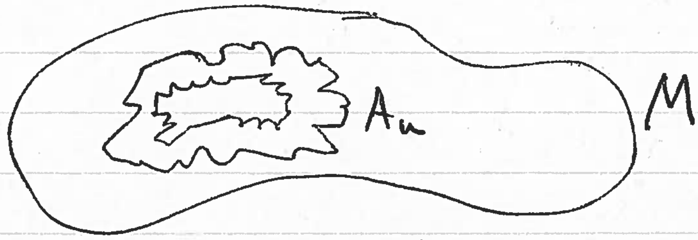
$\phi: M \to \mathbb{R}$ observable (cost function).

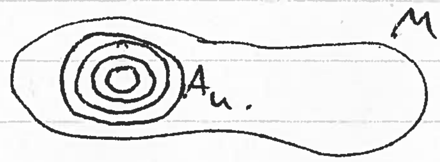
Stochastic processes : $X_n = \phi(f^{n-1}x)$.

So $\{X_n \geq u\} = \{x \in M : \phi(f^{n-1}x) \geq u\}$.

Invariant $\mu \Rightarrow \mu\{X_n \geq u\} = \mu\{X_1 \geq u\}$

$= \mu\{x : \phi(x) \geq u\} := A_u$





<u>Questions</u> :
- What is the frequency of visits to A_u
- If we visit A_u, when do we visit A_u again?
- Do we visit A_u infinitely often.
- What happens if we shrink A_u?

(Replace u by some sequence u_n such that $\mu(A_{u_n}) \to 0$)

<u>Remarks</u>: If $\sum_{n=1}^{\infty} \mu(A_{u_n}) < \infty \Rightarrow P(\limsup A_{u_n}) = 0$

$P(\text{visit } A_{u_n} \text{ i.o.}) = 0$ by BC1.

If $\sum_{n=1}^{\infty} \mu(A_{u_n}) = \infty$ then we need "independence" to ensure $P(\limsup A_{u_n}) = 0$ using BC2.

However this result turns out to be true for reasonable sets

Suppose we can find a_n, u_n and $\tau(u)$ such that

$$n\mu\{x : \phi(x) \geq \frac{u}{a_n} + b_n\} \to \tau(u).$$

Is it true that $\mu\{M_n \leq u_n\} \to e^{-\tau(u)}$

i.e. $\mu\{x \in M : \mathrm{Max}\{\phi(x), \phi(fx), \ldots \phi(f^{n-1}x)\} \leq \frac{u}{a_n} + b_n\}$

$\to e^{-\tau(u)}$?

Answer: Yes: for reasonable (f, M, μ, ϕ).

[Question: Can we compute ~~func~~ functional form of $\tau(u)$?]

What do we need to check?

[H1] Mixing: f is mixing if for all reasonable sets A, B:

$$\mu(A \cap f^{-n}B) \xrightarrow[n\to\infty]{} \mu(A)\mu(B).$$

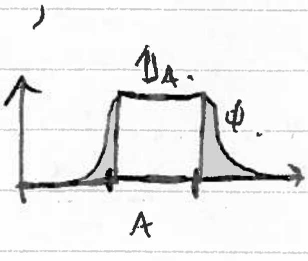
i.e. $\mu(x : x \in A, f^n x \in B) \to \mu(A)\mu(B)$.

or $\mu(x : f^n x \in B \mid x \in A) \to \mu(B)$.

To get "rate" of mixing we need to restrict class of sets — or instead look at correlation decay:

Set $\Theta(n) = \left| \int \varphi \cdot \psi \circ f^n \, d\mu - \int \varphi \, d\mu \int \psi \, d\mu \right|$,

for $\varphi, \psi : M \to \mathbb{R}$ Lipschitz continuous.

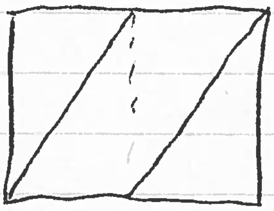


(f, μ, M) has "good" mixing rate if $\exists C > 0, \exists \alpha > 0$ such that $\Theta(n) \leq \frac{C}{n^\alpha}$, $C \equiv C(\varphi, \psi)$.

Examples

- $(f, [0,1], \mathrm{Leb})$

  $f(x) = 2x \bmod 1$

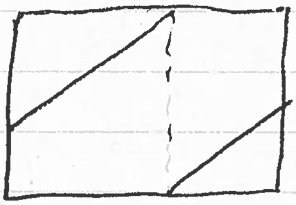


$H(n) \le C r^n$
some $r < 1$.

- $(f, [0,1], \mathrm{Leb})$

  $f(x) = x + \alpha \bmod 1$

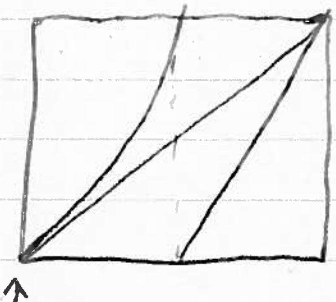
  $\alpha \notin \mathbb{Q}$



$H(n) \not\to 0$.
Not mixing.

- $(f, [0,1], \mu)$.

$$f(x) = \begin{cases} x(1 + 2^\alpha x^\alpha), & x \le \frac{1}{2} \\ 2x - 1 & x > \frac{1}{2}. \end{cases}$$



Tangency.

Here: $\dfrac{d\mu}{dx} \approx \dfrac{C}{x^\alpha}$, $H(n) \approx n^{\frac{1}{\alpha} - 1}$. (so need $\alpha < 1$).

- $(f, \mathbb{R}^2, \mu)$:

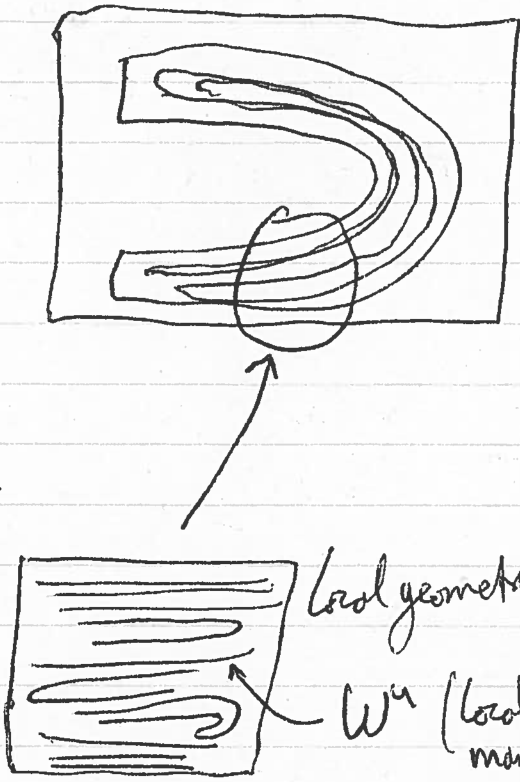
$$f(x,y) = \begin{pmatrix} 1 - a x^2 + y \\ b x \end{pmatrix}$$

$H(n) \le C r^n$, some $r < 1$.

$\mu \in SRB$.

$\mu | W^u$ is absolutely continuous.

$\mathrm{Leb}(\Lambda) = 0$, $\Lambda = \bigcap_{n=0}^{\infty} f^n([0,1]^2)$.



Local geometry.

W^u (local manifold).

- Regularity of μ:

[H2] for an expanding system (non-uniformly expanding), we need $\frac{d\mu}{dx} := \rho(x) \in L^p$ for some $p > 1$.

If μ is SRB we need the local dimension to exist μ-a.e $x \in M$, ie $\exists d : \lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} = d$. ($d \equiv d(x)$ but usually indpt of x)

[H3] • Recurrence control:
Given $g : \mathbb{N} \to \mathbb{R}$ let $E_n = \{x : d(x, f^j x) \leq \frac{1}{n}$, for some $j \leq g(n)\}$

We don't want close returns too soon (typically).
so Given $B \in (0,1)$ let $g(n) = n^B$. Then $\exists\ \tilde{\beta} \in (0,1)$ such that $\mu(E_n) \leq \frac{1}{n^{\tilde{\beta}}}$. [$B$ close to 1 gives $\tilde{\beta}$ close to 0 in practice].

Theorem (HNT, 2010). Suppose (H1)–(H3) hold for sufficiently large p, α. Suppose $\phi(x) = \Phi(\mathrm{dist}(x, \tilde{x}))$, for $\tilde{x} \in M$ and $\Phi : \mathbb{R}^+ \to \mathbb{R}$.

Then for μ a.e $\tilde{x} \in M$, $n\mu\{\phi(x) \geq u_n\} \to \tau(u)$

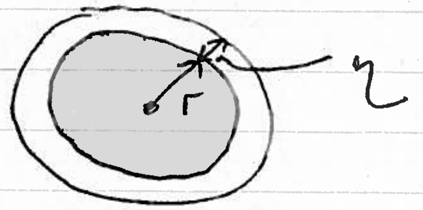
$$\Longrightarrow \quad \mu\{M_n \leq u_n\} \to e^{-\tau(u)}.$$

[Stated for non-unif expanding system].

PROBLEMS : • Checking (H3)
• Finding $\tau(u)$, especially if $\mu \in$ SRB and $\{\phi(x) = u\}$ has complicated geometry.

- For SRB we also need condition (H2b):

$$\exists \gamma>0,\ \forall r<1,\ \forall \eta<r:$$
$$|\mu(B(x,r+\eta)) - \mu(B(x,r))| \le \eta^{\gamma}$$



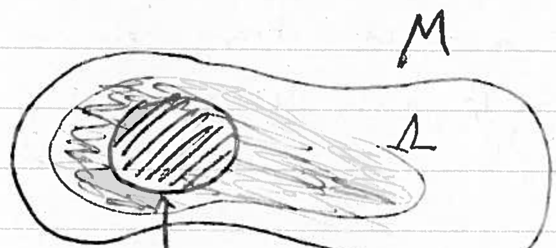
$B(x,r)$: Ball of radius r.

A version of Thm is in progress for the SRB case.

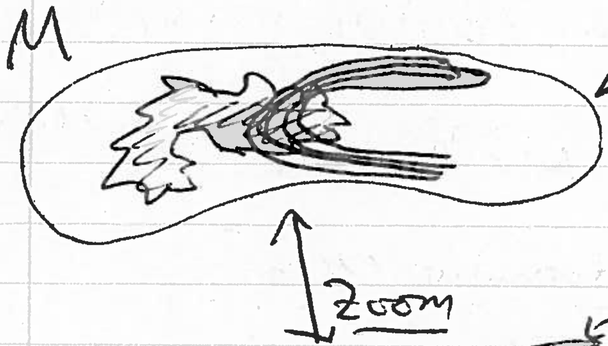
- Geometry:

$A_u = \{\phi(x) \ge u\}$:

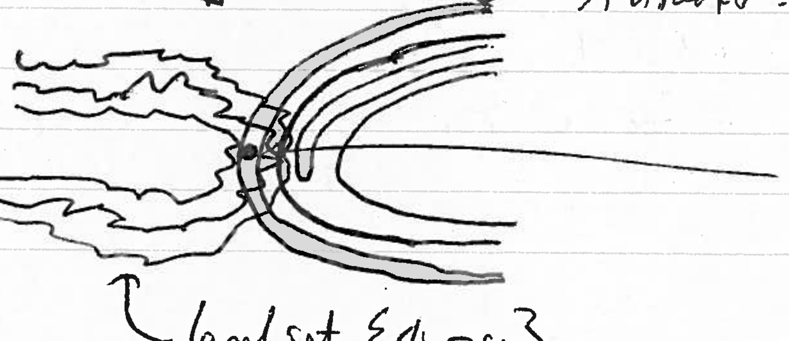
To get $\tau(u)$, we need control on $\mu(A_u)$.



$\mu(A_u)$ easy to compute if (attractor) Λ has trivial geometry, and/or $\{\phi(x)\ge u\}$ are conformal to balls.



Problems if Λ is fractal and level regions $\{\phi(x) \ge u\}$ are not conformal to balls.



Attractor.

$\tilde{x}$: $\phi(\tilde{x})$ takes max value.

level set $\{\phi = u\}$.