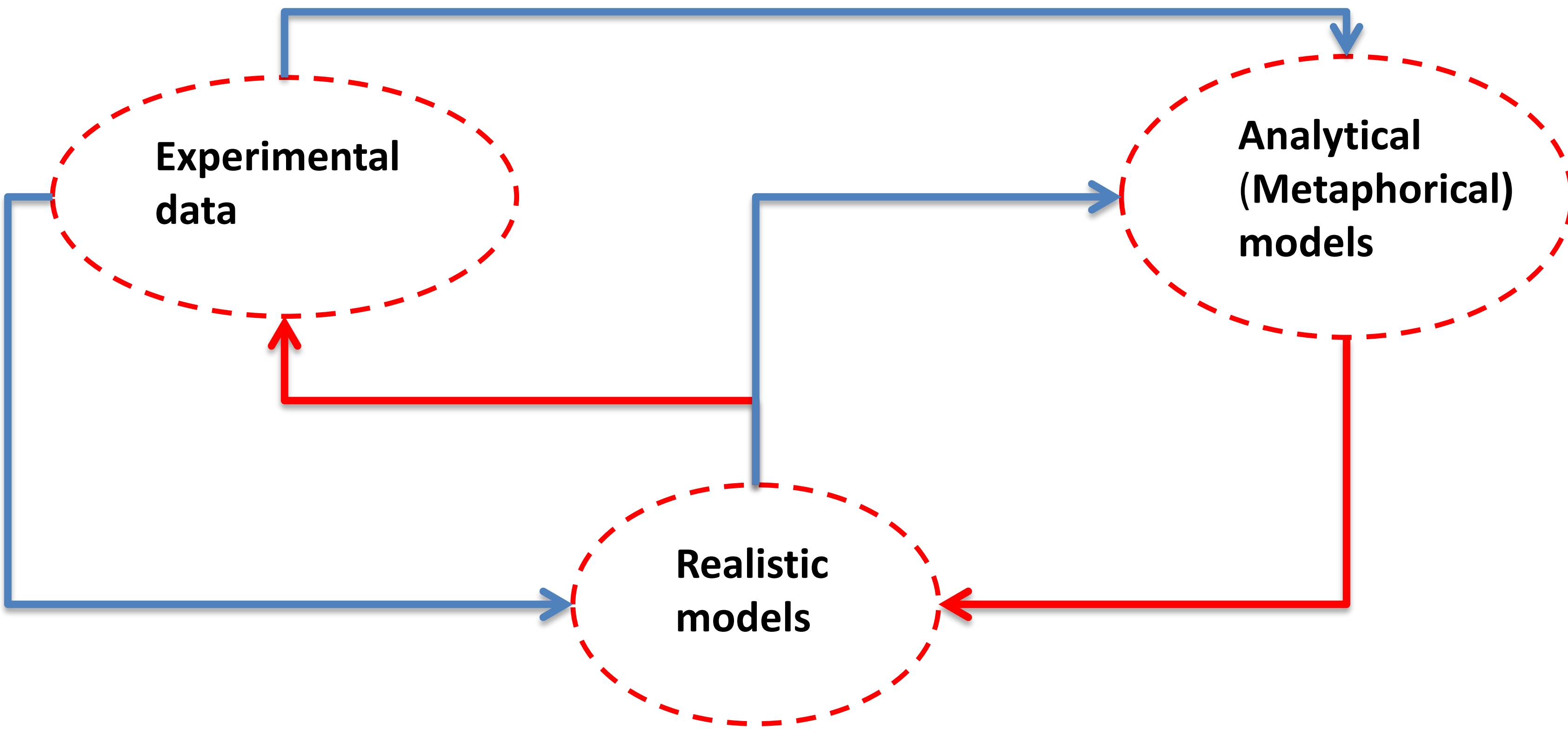
Multiple oscillatory states in models of collective neuronal dynamics

stability, and the role of connections (networks)

22.10.2013

What is the place of Analytical Models?



Class models Z to the even power

$$\frac{d}{dt}Z = (a_n|Z|^{2n} + a_{n-1}|Z|^{2n-2} + \cdots + a_2|Z|^4 + a_1|Z|^2 + A_0)Z + \varepsilon(t)$$

$Z = x + iy$ is complex variable!

$a_n, a_{n-1}, \ldots a_2, a_1$ are real constant coefficients!

$A_0 = c + i\omega$ is a constant complex coefficient!

$\varepsilon(t)$ is the complex input to the system, eventually including white noise components!

What we could do analytically? Some remarks and ideas.

$$\frac{d}{dt}Z = (a_n|Z|^{2n} + a_{n-1}|Z|^{2n-2} + \cdots + a_2|Z|^4 + a_1|Z|^2 + A_0)Z + \varepsilon(t)$$

$$Z \to e^{i\varphi}Z$$

Our equation is invariant under constant phase rotations !

From our equation we can derive two equations describing the time evolution of the radial and angular components!

To analyze the stationary behavior of the solutions of the equation, one may ignore the input $\varepsilon(t)$!

Writing our equation without $\varepsilon(t)$ for variable Z, and the conjugate equation, we are receiving two equations:

What we could do analytically?

$$\frac{d}{dt}Z = (a_n|Z|^{2n} + a_{n-1}|Z|^{2n-2} + \cdots + a_2|Z|^4 + a_1|Z|^2 + A_0)Z + \varepsilon(t)$$

$$\frac{d}{dt}Z = (a_n|Z|^{2n} + a_{n-1}|Z|^{2n-2} + \cdots + a_2|Z|^4 + a_1|Z|^2 + c + i\omega\,)Z$$

$$\frac{d}{dt}\overline{Z} = (a_n|Z|^{2n} + a_{n-1}|Z|^{2n-2} + \cdots + a_2|Z|^4 + a_1|Z|^2 + c - i\omega\,)\overline{Z}$$

$$\overline{Z}\frac{d}{dt}Z = (a_n|Z|^{2n} + a_{n-1}|Z|^{2n-2} + \cdots + a_2|Z|^4 + a_1|Z|^2 + c + i\omega\,)Z\overline{Z}$$

$$Z\frac{d}{dt}\overline{Z} = (a_n|Z|^{2n} + a_{n-1}|Z|^{2n-2} + \cdots + a_2|Z|^4 + a_1|Z|^2 + c - i\omega\,)\overline{Z}Z$$

What we could do analytically? Real part

$$\overline{Z}\frac{d}{dt}Z = (a_n|Z|^{2n} + a_{n-1}|Z|^{2n-2} + \cdots + a_2|Z|^4 + a_1|Z|^2 + c + i\omega\,)Z\overline{Z}$$

$$Z\frac{d}{dt}\overline{Z} = (a_n|Z|^{2n} + a_{n-1}|Z|^{2n-2} + \cdots + a_2|Z|^4 + a_1|Z|^2 + c - i\omega\,)\overline{Z}Z$$

$$Z(t) \equiv \sqrt{\rho(t)}e^{i\varphi(t)}; \rho \equiv |Z|^2 \equiv Z\overline{Z}$$

Real part:

$$\overline{Z}\frac{d}{dt}Z + Z\frac{d}{dt}\overline{Z} = 2(a_n|Z|^{2n} + a_{n-1}|Z|^{2n-2} + \cdots + a_2|Z|^4 + a_1|Z|^2 + c)Z\overline{Z}$$

$$\frac{d}{dt}\rho = 2\rho(a_n\rho^n + a_{n-1}\rho^{n-1} + \cdots + a_2\rho^2 + a_1\rho^1 + c)$$

$$F(\rho) = 2\rho(a_n\rho^n + a_{n-1}\rho^{n-1} + \cdots + a_2\rho^2 + a_1\rho^1 + c)$$

What we could do analytically? Real part

$$\frac{d}{dt}\rho = 2\rho(a_n\rho^n + a_{n-1}\rho^{n-1} + \cdots + a_2\rho^2 + a_1\rho^1 + c)$$

$$F(\rho) = 2\rho(a_n\rho^n + a_{n-1}\rho^{n-1} + \cdots + a_2\rho^2 + a_1\rho^1 + c)$$

The stationary solutions of the first equation, which is $F(\rho) = 0$ corresponds to either a steady state at $\rho = 0$ or to limit cycles for $\rho > 0$ if such solutions exist!

$$\rho(a_n\rho^n + a_{n-1}\rho^{n-1} + \cdots + a_2\rho^2 + a_1\rho^1 + c) = 0$$

Stability of a stationary solution is given by the condition:

$$\frac{d}{d\rho}F(\rho) < 0$$

$$\mu(\rho) \equiv \left[\frac{d}{d\rho}F(\rho)\right]_{F(\rho)=0} < 0;$$

What we could do analytically? Imaginary part

$$\overline{Z}\frac{d}{dt}Z = (a_n|Z|^{2n} + a_{n-1}|Z|^{2n-2} + \dots + a_2|Z|^4 + a_1|Z|^2 + c + i\omega\,)Z\overline{Z}$$

$$Z\frac{d}{dt}\overline{Z} = (a_n|Z|^{2n} + a_{n-1}|Z|^{2n-2} + \dots + a_2|Z|^4 + a_1|Z|^2 + c - i\omega\,)\overline{Z}Z$$

$$Z(t) \equiv \sqrt{\rho(t)}e^{i\varphi(t)}; \rho \equiv |Z|^2 \equiv Z\overline{Z}$$

Imaginary part: $\overline{Z}\frac{d}{dt}Z - Z\frac{d}{dt}\overline{Z} = 2i\rho\omega$

Calculating the derivatives in the left side:

$$\overline{Z}\frac{d}{dt}Z - Z\frac{d}{dt}\overline{Z} = \sqrt{\rho(t)}e^{-i\varphi(t)}\frac{d}{dt}\left(\sqrt{\rho(t)}e^{i\varphi(t)}\right) - \sqrt{\rho(t)}e^{i\varphi(t)}\frac{d}{dt}\left(\sqrt{\rho(t)}e^{-i\varphi(t)}\right)$$

Finally:

$$\overline{Z}\frac{d}{dt}Z - Z\frac{d}{dt}\overline{Z} \equiv 2i\rho\frac{d}{dt}\varphi = 2i\rho\omega$$

$$\frac{d}{dt}\varphi = \omega$$

The last equation shows that the phase velocity, or rotational frequency of the system is given (for $\rho > 0$) by ω, - the imaginary part of the A_0 coefficient!

Application - Model Z to the fourth

$$\frac{d}{dt}Z = (a_1|Z|^2 + A_0)Z;$$
$$A_0 = c + i\omega;$$

$$Z(t) \equiv \sqrt{\rho(t)}e^{i\varphi(t)}; \rho \equiv |Z|^2 \equiv Z\overline{Z};$$

$$F(\rho) = 2\rho(a_1\rho + c);$$
$$\frac{d}{dt}\varphi = \omega$$

$$\rho: F(\rho) = 0 \Rightarrow 2\rho(a_1\rho + c) = 0$$
$$\Rightarrow \ \rho = 0; \ \rho = -c/a_1;$$

$$\mu \equiv \frac{d}{d\rho}F(\rho) < 0 \Rightarrow 4a_1\rho + 2c < 0;$$

$$\{\rho = 0, \mu = c\}$$
$$\{\rho = -\frac{c}{a_1}, \mu = -c\}$$

$$\{a_1, c\}$$

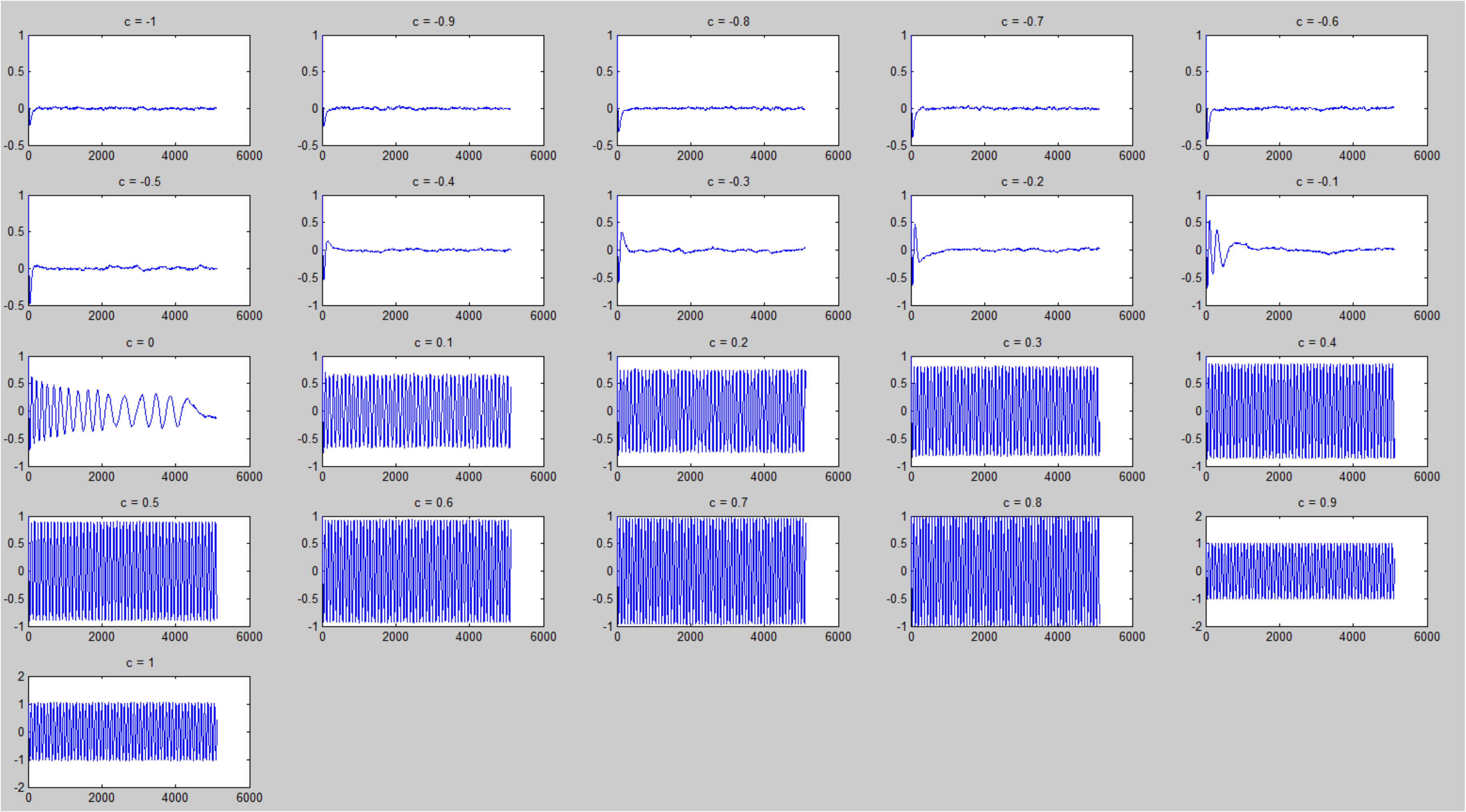
$a_1 > 0 \rightarrow$ *unstable system*

$a_1 < 0, c > 0 \rightarrow$ *one stable limit cycle (LC)*

$a_1 < 0, c < 0 \rightarrow$ *one stable steady state (SS)*

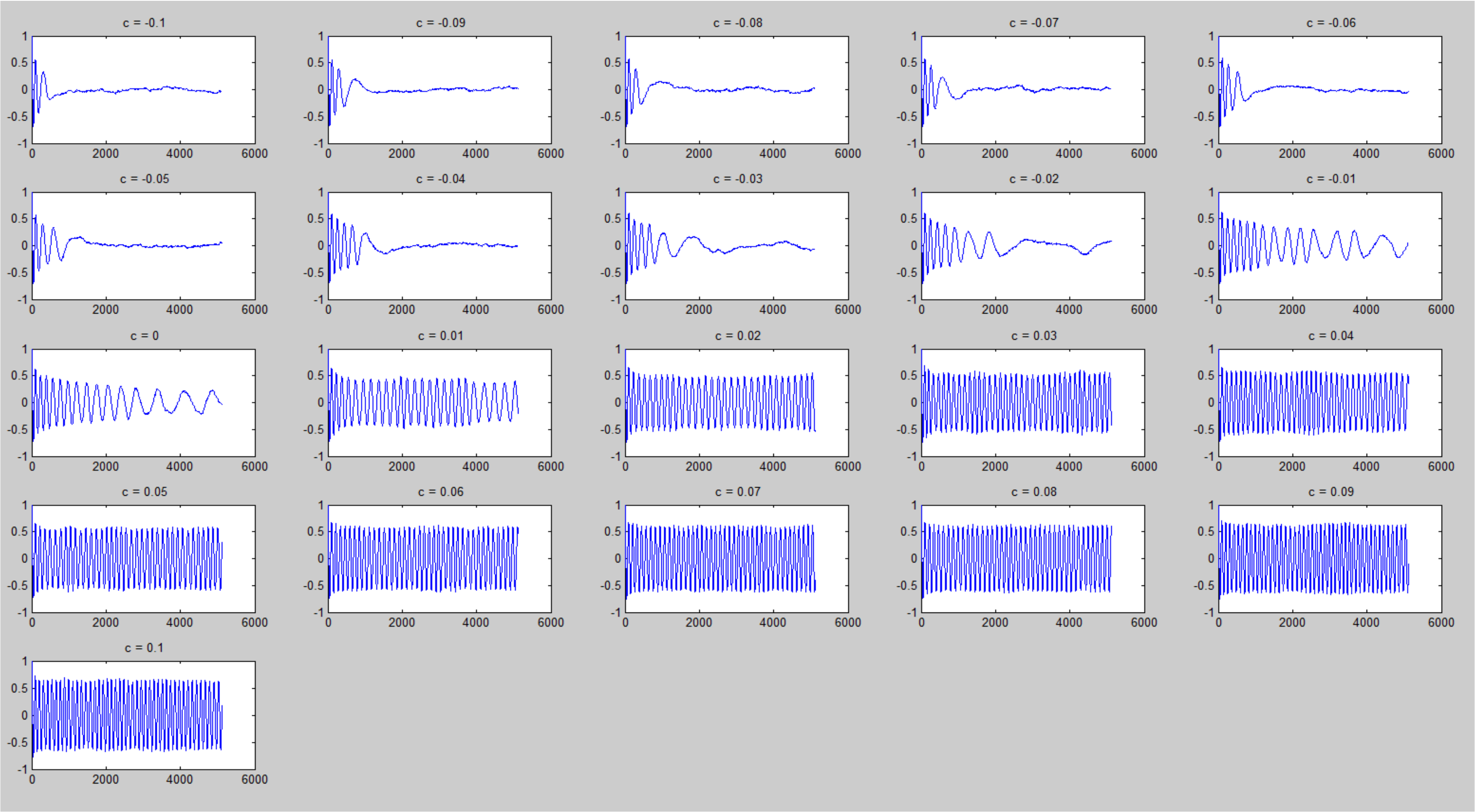
Application – single unit

$a_1 = -1, \ \ -1 \le c \le 1$



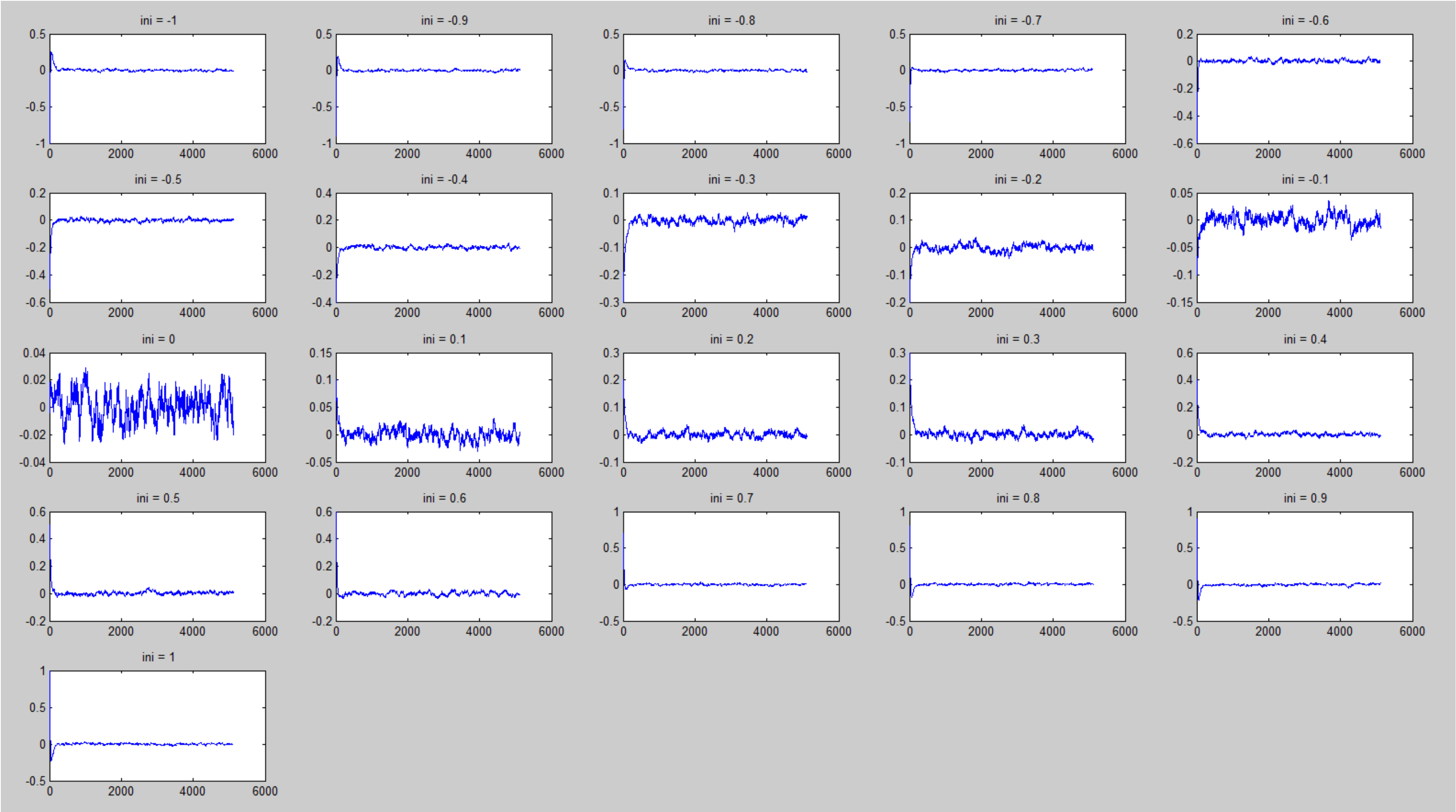
Application – single unit when *c* is close to zero

$a_1 = -1, \ \ -0.1 \le c \le 0.1$



Application – single unit - Dependence on Initial Conditions

$a_1 = -1, c = -0.9 \quad -1 < ini < 1$



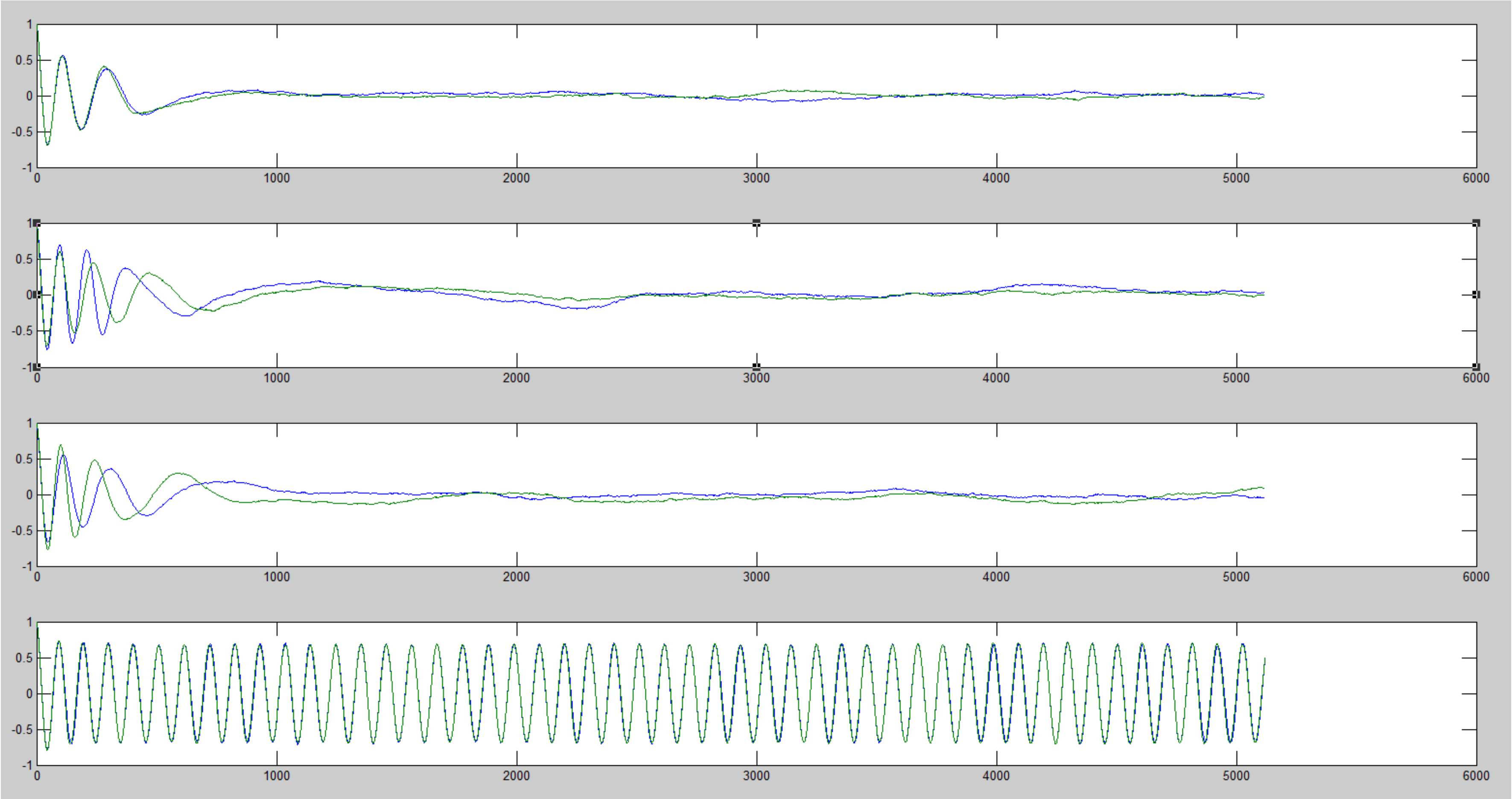
Network Model

$$\frac{d}{dt}Z_i = (a_1|Z_i|^2 + c + i\omega)Z_i + \sum_{j=1}^{N} G_{ij}Z_j + \varepsilon_i(t)$$

Application – two coupled units

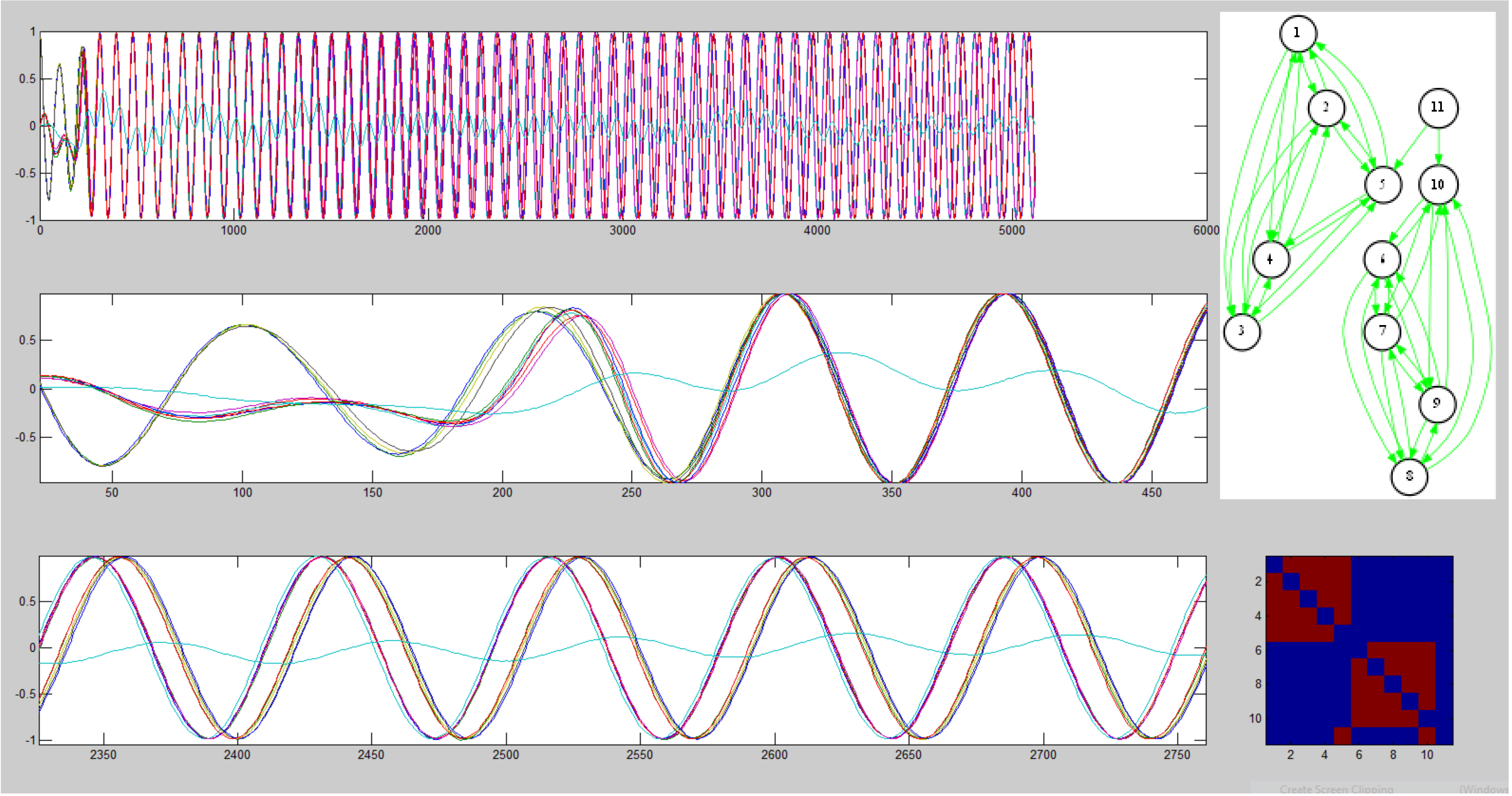
$a_1 = -1, c = -0.09$

1-no conn; 2-blue is conn. to green; 3-green is conn. to blue; 4-bidirectional conn.

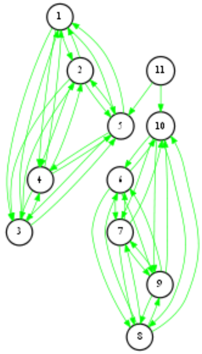


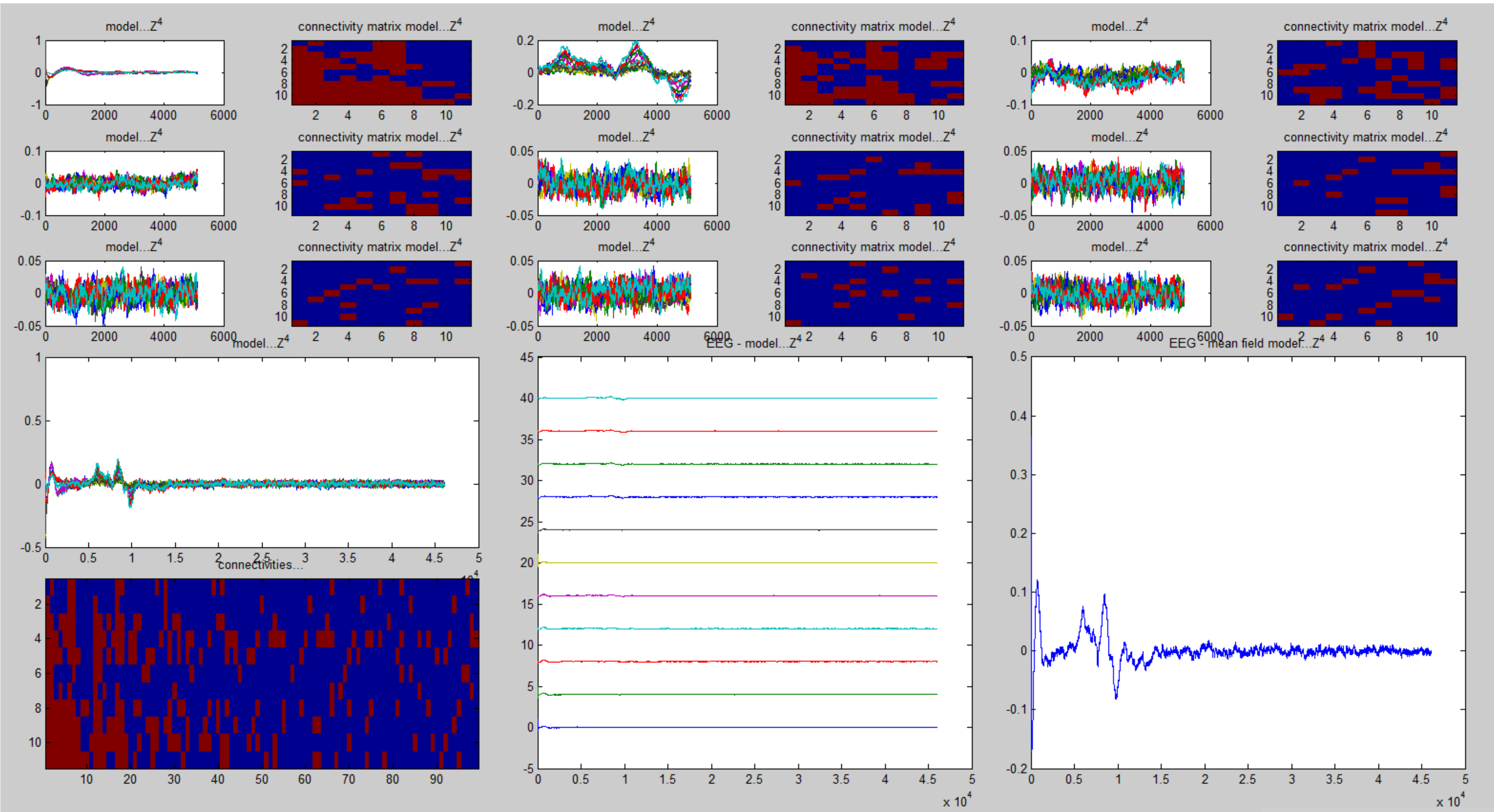
Application – 11 coupled units

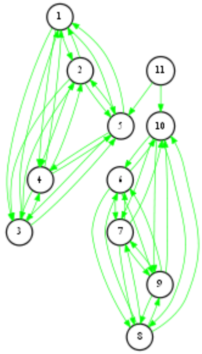
$a_1 = -1, c > -0.09$



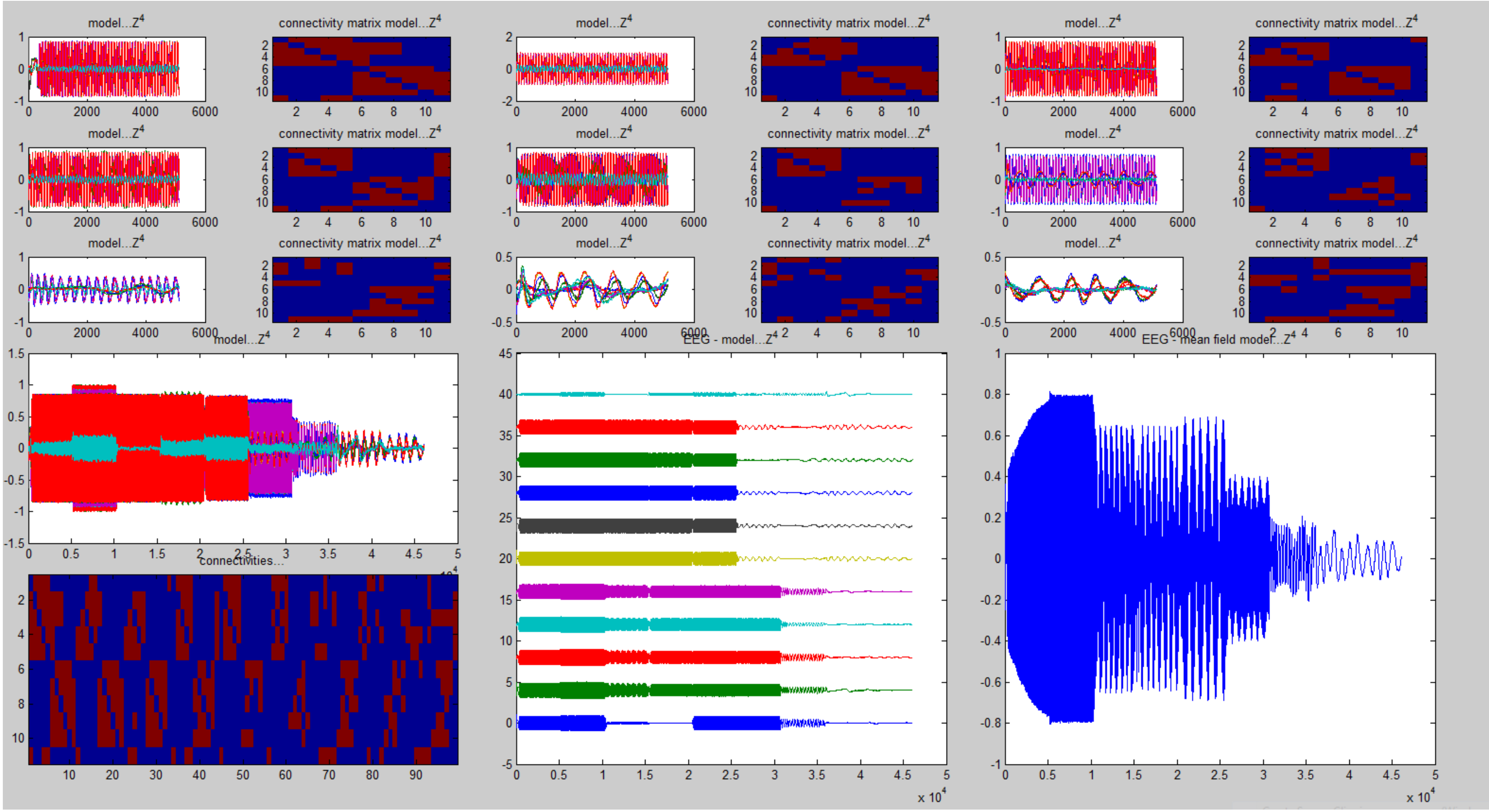
Application (c=-0.9)
11 coupled units
Network Dependence

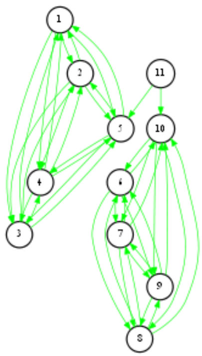






Application (c=-0.5)





Application (c=-0.1)