Equation $\dot{x}(t) = f(x(t), x(t-\tau), p)$, $x(t) \in \mathbb{R}^n$, $p \in \mathbb{R}^k$, $\tau \geq 0$

Symmetry $e^{A\vartheta} f(x, y, p) = f(e^{A\vartheta} x, e^{A\vartheta} y, p)$, $A \in \mathbb{R}^{n \times n}$, $e^{A \cdot 2\pi} = I$, $A^T = -A$

R.Eq. solutions of type $x(t) = e^{A\omega t} x_0$ for all $t \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$, $\omega \in \mathbb{R}$

R.P.O. satisfy $x(t+T) = e^{A\vartheta} x(t)$ for all $t \in \mathbb{R}$, $\vartheta \in \mathbb{R}$, $T \in \mathbb{R}$

Defining equations

$$\underbrace{\begin{array}{ll} 0 = A\omega x_0 - f(x_0, e^{-A\omega\tau} x_0, p) & n\,\text{Eq} \\ 0 = x_r^T A x_0 & 1\,\text{Eq} \\ n+1 \text{ Vars: } x_0, \omega & \end{array}}_{\text{Algebraic system for RE}}$$

$$\underbrace{\begin{array}{ll} \dot{x}(t) = -AT\omega x(t) + T f(x(t), e^{-A\omega\tau} x(t - \tfrac{\tau}{T}), p) & \\ \qquad \text{mod } [0,1] & \\ x(0) - x(1) = 0 & n\,\text{B.C} \\ \int_0^1 x_r(t)^T A x(t)\, dt = 0 & 1\,\text{Eq} \\ \int_0^1 x_r'(t)^T x(t)\, dt = 0 & 1\,\text{Eq} \\ \text{Vars: } x(\cdot) \in C([0,1], \mathbb{R}^n), \omega, T & \end{array}}_{\text{periodic BVP on } [0,1] \text{ for R.P.O.}}$$

Fold of RE:

$$\underbrace{\left.\begin{aligned} 0 &= A\omega x_0 - f(x_0, e^{-A\omega\tau}x_0, p) \\ 0 &= x_r^T A x_0 \end{aligned}\right\}}_{RE} \quad 0 = F(X,p) \qquad X = (x_0,\omega)$$

$$\left.\begin{aligned} 0 &= \partial_X F(X,p)\, y \qquad y \in \mathbb{R}^{n+1} \\ 0 &= y^T y - 1 \end{aligned}\right\} \; 2n+2 \text{ eqs}$$

Vars: $x_0 \in \mathbb{R}^n, \omega \in \mathbb{R}, y \in \mathbb{R}^{n+1}, p \in \mathbb{R}^k$, $k=2 \to$ curve

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Example: Lang-Kobayashi-Eq.

$E(t) \in \mathbb{C} = \mathbb{R}^2 \leadsto A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$n(t) \in \mathbb{R}$

parameters

$$\dot{E}(t) = (1+\alpha)n(t)E(t) + \eta e^{i\varphi} E(t-\tau)$$

$$\dot{n}(t) = \varepsilon \cdot \left(I - n(t) - (2n(t)+1)\overline{E(t)}^T E(t)\right)$$

Basic Eq.: $E=0$, $n=I$

RE: $E(t) = E_0 e^{i\omega t}$ $(|E(t)| = \text{const})$, $n = n_0$

RPO: $|E(t)|$, $n(t)$ periodic