CSDC Reading Group,
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Time-delayed feedback

Given $\dot{x} = f(x, u)$, $x(t) \in \mathbb{R}^n$ state, $u(t) \in \mathbb{R}$ input,

assume system has periodic orbit $x_*(t) = x_*(t-T)$ period T.

(1) Time-delayed feedback control: $u(t) = K^T[x(t-T) - x(t)]$ ← does not use x_* (control gain: K^T)

(2) compare to classical feedback control: $u(t) = K^T[x_*(t) - x(t)]$ ← uses x_*

Problem: find K s.t. $x_*(t)$ is stable

(1) not known

(2) is well known.

Intermediate case: (3)
$$u(t) = K^T[\tilde{x}(t) - x(t)]$$
$$\tilde{x}(t) = (1-\varepsilon)\tilde{x}(t-T) + \varepsilon x(t-T)$$
} extended TDFC. $0 < \varepsilon < 1$

(1) $\varepsilon = 0$, (2) $\varepsilon = 1$

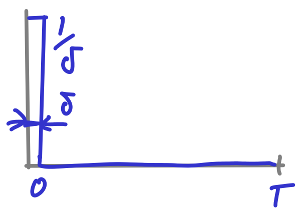
(1) & (3) find unknown periodic orbits

Main result: define $P_0 \in \mathbb{R}^{n \times n}$ monodromy matrix

$P_0 = x(T)$ where $\dot{x}(t) = A(t)x(t)$, $x(0) = I$, $A(t) = \partial_x f(x_*(t), 0)$

$b_0 := \partial_u f(x_*(0), 0) \in \mathbb{R}^n$

$$\Delta_\delta(t) := \begin{cases} \frac{1}{\delta} & \text{if } t \in [0, \delta] \\ 0 & \text{else} \end{cases}$$

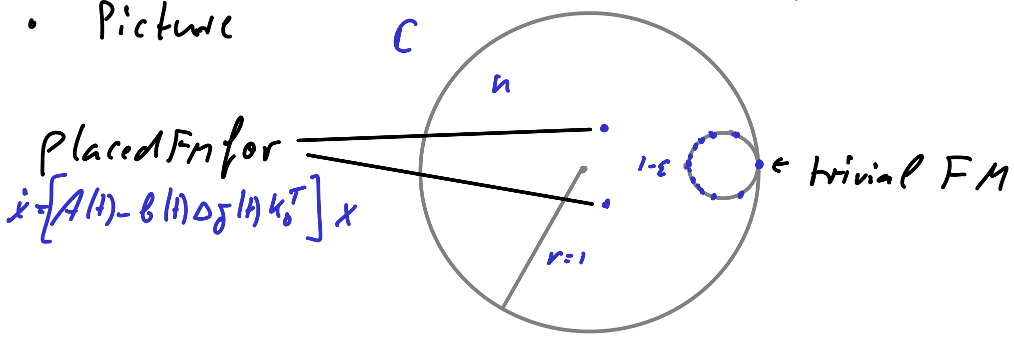


If (P_0, b_0) is controllable ($[b_0, P_0 b_0, \ldots, P_0^{n-1} b_0]$ regular)
then there exists $K_0 \in \mathbb{R}^n$, $\varepsilon_{max} > 0$, $\delta_{max} > 0$
(3) stabilises x_* for $K(t) = \Delta_\delta(t) K_0$ for all $\varepsilon \in (0, \varepsilon_{max})$, $\delta \in (0, \delta_{max})$.

In
$$\dot{x}(t) = f(x(t), \Delta_\delta(t) K^T[\tilde{x}(t) - x(t)])$$
$$\tilde{x}(t) = (1-\varepsilon)\tilde{x}(t-T) + \varepsilon x(t-T)$$
x_* is stable p.o.

Remarks:

- gains are periodic, general positive result does not exist for (2), so unlikely for (3) for constant k
- Picture



- control is "sharp kick" at $t=0$ (w.l.o.g.)
- relies on: single input ($u \in \mathbb{R}$), trivial FM (but argument linear)
- ETDFC known not to work for periodically forced systems and x_* with odd number of FM > 1

Background: $\dot{x} = Ax + bu$ $\quad A \in \mathbb{R}^{n \times n}, \; b \in \mathbb{R}^n$

eigenvalues of $A + bk^T$ can be placed arbitrarily by choosing k if $[b, Ab, \dots A^{n-1}b]$ reg.

similarly:

$$\dot{x}(t) = f(x(t), \Delta_\delta(t)\, k_0^T (x_*(t) - x(t))) \quad \text{, linearized in } x_*$$

$$\dot{x}(t) = A(t)\, x(t) - b(t)\Delta_\delta(t)\, k_0^T x(t) \qquad (*2)$$

let $P(r,s)x_0$ be monodromy matrix of $\dot{x} = A(t)x$, $x(s) = x_0$, $x(r) = P(r,s)x_0$

$t \in \delta$ $\quad x(t) = P(t)x_0 - \int_0^t \underbrace{P(t,s)}_{I + O(\delta)} \underbrace{b(s)k^T}_{\frac{1}{\delta} b_0 + O(\delta)} x(s)\, ds$

$$x(\delta) = x_0 - \frac{1}{\delta} b(0) k^T \int_0^\delta x(s)\, ds = \dots \; \exp(-b_0 \, k_0^T)\, x_0$$

$\hookrightarrow$ Time-T map of $(*2)$ is $P_0 \exp(-b_0 \, k_0^T) + O(\delta)$

Eigenvalues of $P_0 \exp(-b_0 k_0^T)$ can be placed arbitrarily (up to $\lambda_1 \dots \lambda_n > 0$) if $[b_0, P_0 b_0, \dots P_0^{n-1} b_0]$ reg.

$\Rightarrow$ classical feedback control can place FM.

$$\left(\quad \exp(-b_0 k_0^T \mu) = I - b_0 k_0^T \left(\frac{\exp(\mu\, k_0^T b_0) - 1}{k_0^T b_0} \right) \quad \right)$$

Characteristic equation for FM of delayed feedback

Linearisation: (1) $\dot{x} = A(t)x + b(t)\Delta_\delta(t)k_0^T(\tilde{x}(t) - x(t))$

(2) $\tilde{x}(t) = (1-\varepsilon)\tilde{x}(t-T) + \varepsilon x(t-T)$

$x(t) = \lambda x(t-T)$

$\tilde{x}(t) = \lambda \tilde{x}(t-T)$ ⟶ (2) $[\lambda - (1-\varepsilon)]\tilde{x}(t-T) = \varepsilon x(t-T)$

$$\tilde{x}(t) = \frac{\varepsilon}{\lambda-(1-\varepsilon)} x(t)$$

Insert into (1) $\dot{x} = A(t)x + b(t)\Delta\delta(t)k_0^T\left(\frac{\varepsilon}{\lambda-(1-\varepsilon)} - 1\right)x$

$$\lambda x = P_0 \exp\left[-b_0 k^T\left(1 - \frac{\varepsilon}{\lambda-(1-\varepsilon)}\right)\right]x + O(\delta)x$$

$$h(\lambda,\varepsilon,\delta) = \det\left\{\lambda I - P_0 \exp\left[-b_0 k_0^T\left(1-\frac{\varepsilon}{\lambda-(1-\varepsilon)}\right)\right]\right\} + O(\delta)$$

if λ far away from $1-\varepsilon$ ⟶ $h(\lambda,\varepsilon,0) = \det[\lambda - P_0 \exp(-b_0 k_0^T)] + O(\varepsilon)$

↳ FM if & only if FM for classical feedback control

$\lambda \approx 1-\varepsilon$: $\lambda = 1 - \varepsilon + \frac{\varepsilon}{\kappa}$ ⟶ $h(\kappa,\varepsilon,0) = \det\left[I - P_0 \exp\left(b_0 k_0^T(1-\kappa)\right)\right] + O(\varepsilon)$ ($|\kappa| > c > 0$)

$= \det\left[I - P_0\left(I + b_0 k_0^T\sigma\right)\right]$ $\quad \sigma = \frac{\exp(k^T b_0(\kappa-1)) - 1}{k_0^T b_0}$

$= \det\left[(I - P_0) - b_0 k_0^T \sigma\right]$

($I - P_0$) has rank $n-1$; $b_0 k_0^T$ rank 1

⟶ is linear function in σ

↳ is zero iff $\sigma = 0$

↳ $\exp(k_0^T b_0(\kappa-1)) = 1$

↳ $\kappa_l = 1 + \frac{2\pi i l}{k_0^T b_0}$ ↳ $\lambda = 1 - \varepsilon + \frac{\varepsilon}{\kappa}$ on circle

↑ for $h(\kappa,0,0)$. For $h(\kappa,\varepsilon,0)$ all κ_l with $l \neq 0$ are still $|\kappa_l| > 1$ for small ε. ⟶ roots of $h(\lambda,\varepsilon,0)$ inside unit circle.