Extremes for Energy-Like Observables on Hyperbolic Systems

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Dynamical system as a stochastic process

Definition

Given a dynamical system (T, X, μ) we define a stochastic process

$$X_n = \varphi \circ T^n(x)$$

where $\varphi : X \to \mathbb{R}$ is an observable representing some physical quantity which can be measured and holds some regularity.

In modeling deterministic physical phenomenon, T is usually taken as ergodic and measure-preserving and μ a probability measure.

Extreme values

Definition

Given a sequence of random variables $X_1, ..., X_n$ we define the maxima of the system by,

$$M_n = \max\{X_1, ..., X_n\}$$

- In this setting we can investigate the statistical properties of (M_n) such as distributional and almost sure convergence limits.
- These statistical properties depend on our choice of observable.
- In extreme value literature, $\varphi = f(d(x, p))$ for $x \in X$ and some distinguished point $p \in X$ where f is usually monotone decreasing with $\sup_x \varphi(x) = \varphi(p)$.
- Let $\mathcal{S}$ the set where $\varphi(x)$ reaches its supremum. ($\mathcal{S} = \{p\}$ above)

Extreme Value Law (EVL)

Definition

Let (u_n) be a sequence of constants defined by the requirement that $\lim_{n\to\infty} n\mu(X_1 > u_n) = \tau$ and $X_1, ..., X_n$ be i.i.d random variables then,

$$\lim_{n\to\infty} \mu(M_n \le u_n) = e^{-\theta\tau}$$

where $\theta \in [0,1]$ is called the extremal index where $\frac{1}{\theta}$ roughly measures the clustering of exceedences of the maxima.

Dependence Conditions

Definition (Leadbetter (mixing condition))

Condition $D(u_n)$: Given the sequence $X_1, ..., X_n$, for any integers $i_1 < ... < i_p$ and $j_1, ..., j_k$ for which $j_1 - j_p > t$, and any large $n \in \mathbb{N}$,

$$|F_{i_1,...,i_p,j_1,...,j_k}(u_n) - F_{i_1,...,i_p}(u_n)F_{j_1,...,j_k}(u_n)| \leq \alpha(n,t)$$

uniformly for every $p, k \in \mathbb{N}$, where $F_{i_1,...,i_p}$ denotes the joint distribution function of $X_{i_1}, ..., X_{i_p}$ and $\alpha(n, t_n) \to 0$ as $n \to \infty$ for $t_n = o(n)$.

Definition (Leadbetter (recurrence condition))

Condition $D'(u_n)$: Given the sequence $X_1, ..., X_n$ there exists a sequence k_n such that $k \to \infty$, $\lim_{n\to\infty} k_n \alpha(n, t_n) = 0$ and $k_n t_n = o(n)$ and,

$$\lim_{n\to\infty} \sum_{j=1}^{\lfloor n/k_n \rfloor} \mathbb{P}(X_1 > u_n, X_j > u_n) = 0$$

Previous Work

EVL results for classical observables of the form $\varphi(x) = f(d(x, p))$ for some point $p \in X$

- Under $D(u_n)$ and $D'(u_n)$ an extreme value law exists for non-uniformly expanding maps. (Nicol, Holland, Torok (2012))
- For certain one-dimensional uniformly expanding maps, $\theta = 1$ if p is not periodic and $\theta < 1$ otherwise. (Ferguson, Pollicott (2012))
- Dynamical variations of $D(u_n)$ ($Д(u_n)$) and $D'(u_n)$ ($Д'(u_n)$) were introduced. (Freitas, Freitas, Todd (2010))
- $\theta = 1$ a.e. x for Sinai Dispersing Billiards with non-periodic point p. (Haydn, Freitas, Nicol (2014))
- $\theta < 1$ for Sinai Dispersing Billiards with periodic point p. (C., Nicol, Zhang (2018))

- In extreme value literature $\varphi = -\log d(x, p)$ is often used, however, scaling can translate results for one observable to another provided $\mathcal{S}$ remains unchanged.
- If $\mathcal{S}$ does change, there are no known translation results.
- As we have seen, even in the case when p changes (e.g. p periodic, versus p non-periodic), extreme value results have been shown to change.

Can we extend EVL results to observables with more physical relevance where $\mathcal{S}$ is represented by a curve rather than a point?

Previous Work

- Results for certain Anosov diffeomorphisms for observables of the form,

$$\varphi(x, y) = 1 - |x - x_M|^a - |y - y_M|^b$$

and

$$\varphi(x, y, z) = ax + by + cz + d$$

using the geometry of the level sets $\varphi(x, y) > u_n$ and the geometry of the underlying attractor. (Holland, Vitolo, Rabassa, Sterk, and Broer (2012))

- Observables of the form $\varphi = -\log d(x, L)$ where L is a line were investigated in the setting of two-coupled expanding maps (Keller and Liverani (2009)) and N-coupled expanding maps (Faranda, Ghoudi, Guiraud, and Vaienti (2018)).

Some remarks

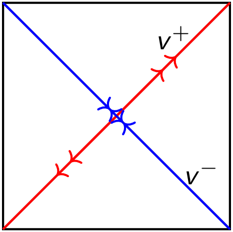
- In order to establish an extreme value law (EVL) for the following systems, we need to show conditions $\text{Д}(u_n)$ and $\text{Д}'(u_n)$ hold.
- Condition $\text{Д}(u_n)$ is a somewhat standard decay of correlations argument and will not be checked here.
- The novelty of these (and many proofs) come from showing $\text{Д}'(u_n)$ holds.
- $\text{Д}'(u_n)$ looks at ensuring,

$$\lim_{n\to\infty} \sum_j \mu(U_n \cap T^j(U_n)) = 0$$

where $U_n = \{x : \varphi(x) > u_n\}$

Suppose that (T, X, μ) is an Anosov system. Further, consider the Arnold Cat Map of $\mathbb{T}^2$ induced by the matrix,

$$T = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$



- This matrix has two eigenvalues $|\lambda_+| > 1$ and $|\lambda_-| < 1$.
- Any $v = \alpha v^+ + \beta v^-$ and $v^{(n)} = DT^n v = \alpha\lambda_+^n + \beta\lambda_-^n$.

Theorem (C., Holland, Nicol)

Let (T, X, μ) *be an Anosov system, and consider the observable function* $\varphi(x) = -\log(|x \cdot v - c|)$ *where* $x = (x_1, x_2) \in \mathbb{R}^2$, $v = (v_1, v_2) \in \mathbb{R}^2$, $c \in \mathbb{R}$. *Then* $\mathcal{S} := \{x \in \mathbb{R}^2 : x \cdot v = c\}$. *We have the following:*

1. *Suppose that* $v \neq \{v^+, v^-\}$. *Then*

$$\lim_{n\to\infty} \mu(M_n \leq u_n) = e^{-\tau}. \tag{1}$$

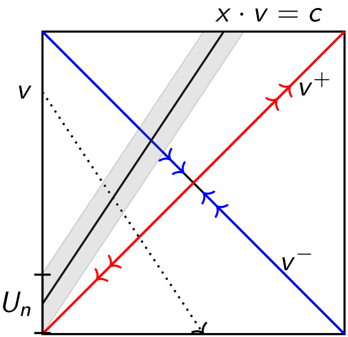
2. *Suppose that* $v = v^+$ *or* $v = v^-$, *and* $\mathcal{S}$ *contains no periodic points. Then equation* (1) *applies.*
3. *Suppose that* $v = v^+$ *or* $v = v^-$, *and* $\mathcal{S}$ *contains a periodic point* p *of prime period* q. *Then,*

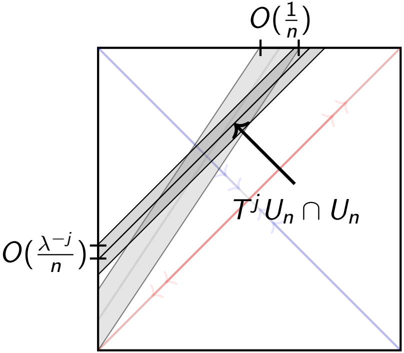
$$\lim_{n\to\infty} \mu(M_n \leq u_n) = e^{-\theta\tau}.$$

where $\theta = 1 - \frac{1}{\lambda_+^q}$.

Checking condition $Д'(u_n)$: case 1.

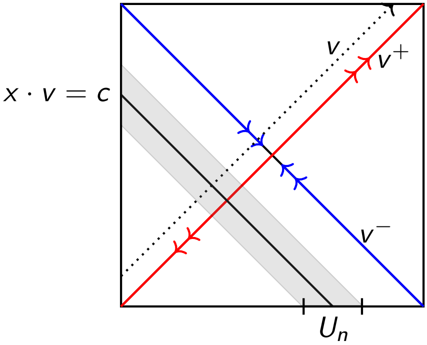
The sets $U_n = \{\varphi(x) > u_n\}$ are defined by a rectangular box around the line defined by $\mathcal{S} := \{x \in \mathbb{R}^2 : x \cdot v = c\}$.
case 1. $\alpha \neq 0$ and $\beta \neq 0$





- There are approximately λ^j intersections of $T^j U_n$ into U_n.
- We estimate $\sum_j \mu(T^j U_n \cap U_n) = O(\lambda^j \cdot \frac{\lambda^{-j}}{n} \frac{1}{n}) = O(\frac{1}{n^2})$.

case 2. a. $\beta = 0$ then v aligns with v^+



- Similar to case 1.
- We will not cover case 2. b. v aligns with v^-.

Checking condition $\amalg'(u_n)$: case 3. (periodic)

- Let p be a periodic point in $\mathcal{S}$ of prime period q.
- $A_n^q = \{\varphi > u_n, \varphi \circ T < u_n, \ldots, \varphi \circ T^q < u_n\}$
- Geometrically, A_n^q consists of two small outer parallel strips in U_n of width $(1/n)(\frac{1}{\lambda_+^q})$.
- The proof of is the same as in the case of no periodic orbits where A_n^q plays the role of U_n.
- By definition,

$$\theta = \lim_{n\to\infty} \frac{\mu(A_n^q)}{\mu(U_n)} = 1 - \frac{1}{\lambda_+^q}$$

Sinai dispersing billiard model with finite horizon

Sinai Dispersing Billiard Animation

Sinai dispersing billiard model with finite horizon

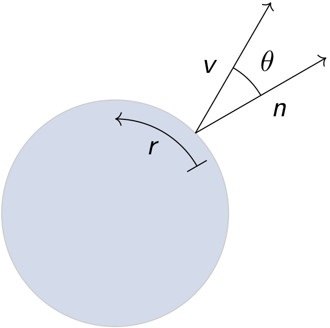


Illustration of the reduction to the billiard map for a single collision point.

Theorem (C., Holland, Nicol)

Let $T : X \to X$ by a planar dispersing billiard map with finite time horizon. Suppose $\varphi(r,\theta) = 1 - |r - r_0|$ where $x = (r,\theta)$ gives $\varphi(x) = 1 - d_H(x, L)$. Assume L is not in the stable or unstable cone. Then,

$$\lim_{n\to\infty} \mu(M_n \leq u_n) \to e^{-\tau}$$

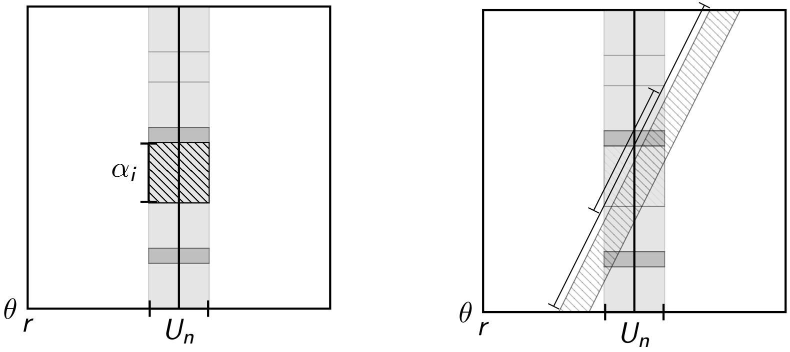
Remark

With this choice of φ the set $\mathcal{S}$ is a line so that $U_n = \{x : \varphi(x) > u_n\}$ forms a rectangle around $\mathcal{S}$. The hyperbolic properties of the billiard map make this similar to the Anosov case with two main difficulties: non-uniform expansion and the presence of singularities in the space.

- Chaotic Billiards (Chernov, Markarian (2006))

Checking condition Д$'(u_n)$

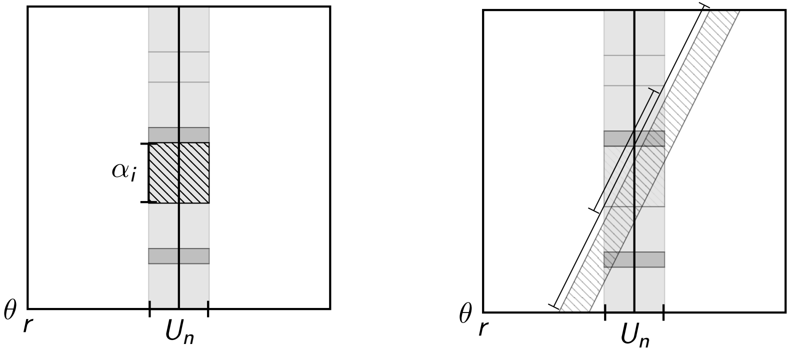
short returns



- Consider the set of all points on X which will not hit a singularity in $j = C \log n$ iterates.
- Map these backward and look at their intersection with our line $r = r_0$.
- All rectangles R_i with side length $\alpha_i < \frac{1}{\sqrt{n}}$ are ignored since any intersection with them decays *quickly*.

Checking condition $Д'(u_n)$

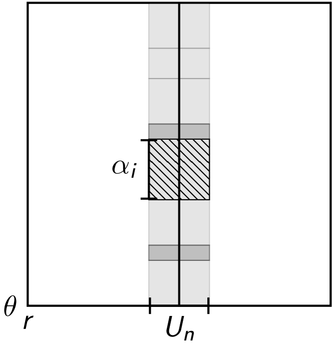
short returns

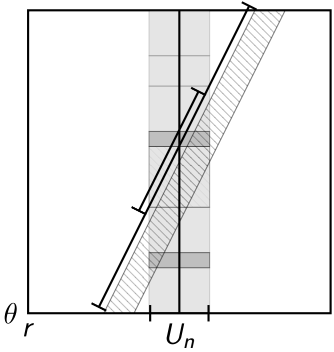


- Any set inside U_n can intersect U_n at most one time by $N^j/n = 1$ and solving for j.
- We estimate the portion of $T^j R_i$ that intersects U_n by $O(n^{-1})/O(\alpha_i)$.

Checking condition $Д'(u_n)$

short returns





- Since $\alpha_i \geq \frac{1}{\sqrt{n}}$ we estimate, $\mu(T^j R_i \cap U_n) = o(\frac{n^{-1}}{n^{-1/2}}\mu(R_i))$ and hence,

$$\sum_{R_i} \mu(U_n \cap T^j R_i) \leq Cn^{-1/2}\mu(R_i)\mu(U_n) \leq Cn^{-5/4}\mu(U_n).$$

Checking condition $\text{Д}'(u_n)$

intermediate returns

- **Dealing with non-uniform expansion.** Define a set of line segments where a local unstable (stable) manifold is homogeneous (has uniform expansion rates) if it does not intersect the line segments. (Chernov)
- **Dealing with singularities. Fragmentation of the phase space into U_n.**
  - Define $\gamma_n(x) = W^u(x) \cap U_n$ and note that $T^j \gamma_n(x)$ consists of a connected curve for $j < C \log n$ iterates.
  - If $T^{i+j}\gamma_n(x)$ intersects a singularity line then it breaks into a set of connected components V_n.
  - We use one-step expansion to obtain bounds on the set of V_n which are small and hence, may remain in U_n for a *long time*.
  - This set is shown to decay *quick enough* to zero as $n \to \infty$.

Coupled systems of uniformly expanding maps

Define the following coupled system of uniformly expanding maps T of the interval,

$$F(x_i) = ((1-\gamma)Tx_i, \frac{\gamma}{N}\sum_j Tx_j)$$

with observable of the form $\varphi(p) = -log(||p^{\perp}||)$ where the component of $p = (x_1, x_2, \ldots, x_N)$ orthogonal to the line $L = \{(x_1, x_2, \ldots, x_N) : x_1 = x_2 = \cdots = x_N\}$ is $p^{\perp} = (x_1 - \bar{x}, x_2 - \bar{x}, \ldots, x_N - \bar{x})$ where $\bar{x} = \frac{1}{N}\sum_{j=1}^{N} x_j$.

Remark

Here $\mathcal{S}$ is the line $x_1 = x_2 = \cdots = x_N$.

Coupled systems of uniformly expanding maps

Geometric interpretation of the observable.

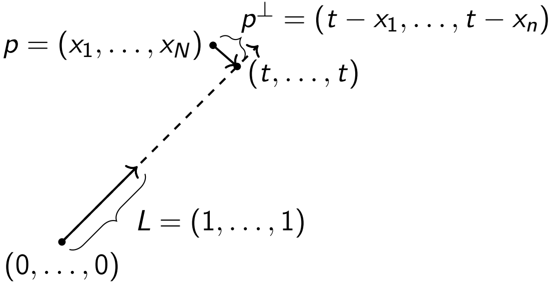


Figure: The distance is given by the magnitude of the vector $p^\perp$.

Previous work

- Results for the existence of an EVL and the value of the extremal index (EI) in the case of a two coupled system where the averaged term is given by $\frac{\gamma}{N}\sum_{j\neq i} Tx_j$ were obtained by Keller and Liverani (2009).
- Results for the existence of an EVL and value of the EI in the case of N coupled system using a transfer operator approach is provided by Faranda, Ghoudi, Guiraud, and Vaienti (2018).
- We extend results by Faranda et al. using a pure probabilistic approach.

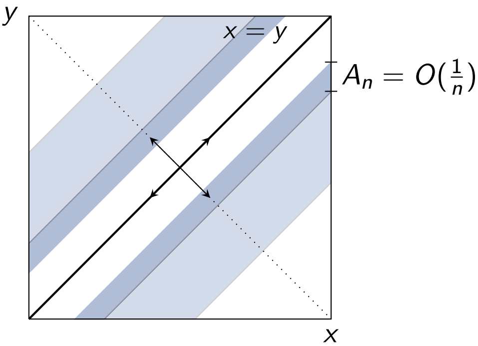
Theorem (C., Holland, Nicol)

Let $F : T^N \to T^N$ be a coupled system of N expanding maps. Suppose $\varphi(p) = -\log(||p^{\perp}||)$, then

$$\lim_{n\to\infty} \mu(M_n \leq u_n) = e^{-\theta\tau}$$

where $\theta = 1 - \frac{1}{(1-\gamma)^{N-1}} \frac{1}{|DT(x)|^{N-1}} \int h(x)dx$.

Checking condition $Д'(u_n)$

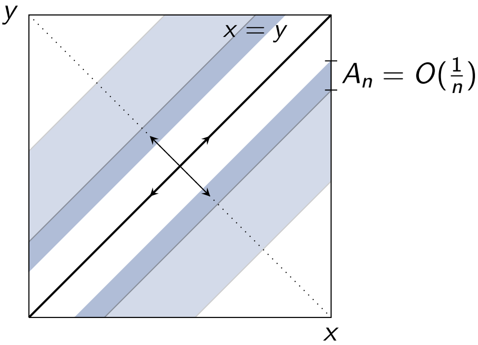


- For simplification, we will only look at the two coupled system

$$F(x, y) = ((1-\gamma)Tx, \frac{\gamma}{2}(Tx + Ty), (1-\gamma)Ty, \frac{\gamma}{2}(Tx + Ty)).$$

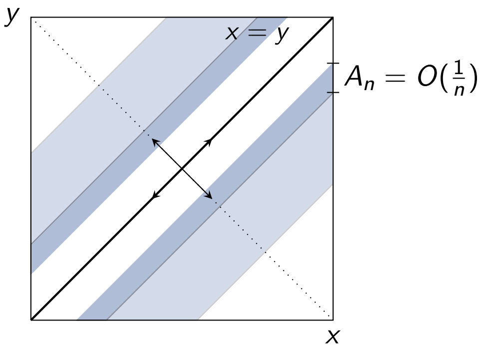
- We define the set $A_n = \{\varphi > u_n, \varphi \circ F < u_n\}$.
- The invariant line $x = y$ is uniformly repelling in all directions.

Checking condition $\beth'(u_n)$



- We use coordinates $v = \frac{x-y}{\sqrt{2}}$ to measure the perpendicular distance to the line.
- For $j = 1, \ldots, C_1 \log n$ this uniform repulsion ensures that $\mu(A_n \cap F^j A_n) = 0$ until $|F^j x - F^j y| = O(1)$.

Checking condition $Д'(u_n)$



- For any expanding map $T = rx$ we have expansion of A_n by the map $F^{[C_1 \log n]}$ given by at least $C_2 r^{[C_1 \log n]} \sim n^{\alpha}$ for some $0 < \alpha < 1$.
- Thus, for $C_1 \log n \leq j \leq C_3 \log n$, $\mu(A_n \cap F^{-j} A_n) \leq \frac{1}{n^{1+\alpha}}$.

Calculating the extremal index for coupling maps

Let $v = \frac{x-y}{\sqrt{2}}$ and $u = \frac{x+y}{\sqrt{2}}$,

$$\theta = \lim_{n\to\infty} \theta_n = \lim_{n\to\infty} \frac{\mu(A_n)}{\mu(U_n)}.$$

But

$$\frac{\mu(A_n)}{\mu(U_n)}$$

$$\sim \lim_{n\to\infty} [1 - \int_0^{\frac{1}{n[Tv]}} h(u,v)\ du\ dv / \int_0^{\frac{1}{n}} h(u,v)\ du\ dv]$$

$$= 1 - \frac{1}{(1-\gamma)} \frac{1}{|DT|} \int h(u) du,$$

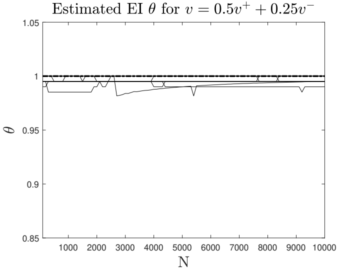
since $v \to Tv$.

Numerically estimating the extremal index

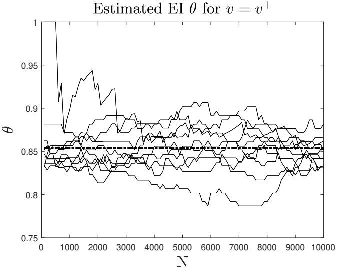
Recall: Definition of the extremal index is the ratio of the number of exceedances in a cluster to the total number of exceedances.

- 'Blocks estimator': splits the data into fixed blocks of size k_n and defines a cluster by the number of exceedances inside a block.
- 'Runs estimator': introduces a run length of q_n so that any two exceedances separated by a time gap of less than q_n belong to the same cluster.
  - Heavy dependence on choice of k_n and q_n. (Lucarini et. al (2014), Extremes Book)
- Consider the point process of exceedances as a Poisson process (under certain regularity conditions).
  - Extremal index as the expected value of a Poisson process.
  - Süveges estimate essentially the log-likelihood estimate of the expected value of the Poisson point process. (Süveges (2007))

Numerical results on the extremal index for the Anosov system



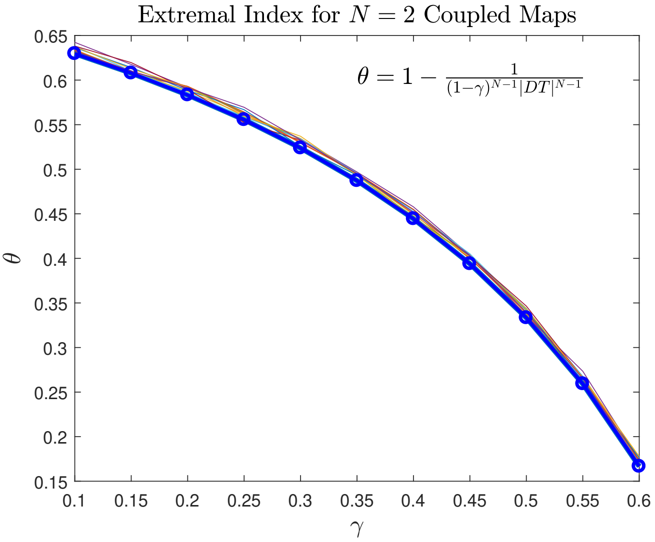
(a)



(b)

Estimated extremal index for increasing samples of the Anosov system for (a) the line L transverse to the stable and unstable directions and (b) the line L inline with the stable direction (v aligns with v^+) with periodic point of period 2.

Numerical results on the extremal index for coupled expanding maps



Numerical results on the extremal index for coupled expanding maps